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This problem set was reviewed by a tutor. *This does not mean that it is a perfect solution. Neither I or the tutor imply that there are no further mistakes in this document.*

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[disclaimer]

physics755 – Quantum Field Theory

Problem Set 12

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Group Tuesday – Ripunjay Acharya

problem	achieved points	possible points
Dimensional regularization of ϕ^4 theory at 1-loop	6	15
total	6	15

This document consists of 8 pages.

We did not get much of the results we should have gotten on this sheet. The tutorial on 2015-07-14 will hopefully have cleared things up. This will serve as a documentation of our non-understanding in retrospect.

I hope so too!

1 Dimensional regularization of ϕ^4 theory at 1-loop

1.1 Substitution

0.5/0.5

We perform the Wick rotation by $p^0 \mapsto \mathrm{i}p^0$.

$$\begin{aligned}\int \frac{\mathrm{d}^d p}{[2\pi]^d} \frac{1}{[p^2 - A]^n} &= \int \frac{\mathrm{d}^d p}{[2\pi]^d} \frac{1}{[[p^0]^2 - \boldsymbol{p}^2 - A]^n} \\ &\mapsto \int \frac{\mathrm{i}\mathrm{d}^d p}{[2\pi]^d} \frac{1}{[-[p^0]^2 - \boldsymbol{p}^2 - A]^n} \\ &= [-1]^n \mathrm{i} \int \frac{\mathrm{d}^d p}{[2\pi]^d} \frac{1}{[[p^0]^2 + \boldsymbol{p}^2 + A]^n}.\end{aligned}$$

✓

One thing Penrose (2005) warns to be careful about is that the Wick rotation can turn a non-compact group like SO(1,3) into a compact group like SO(4). Then one works with the compact group and goes back. One should check whether the results one got in the Euclidean space still makes sense in the Minkowski space.

1.2 Proof of identity

This problem is also somewhat covered by Peskin and Schroeder (1995, pp. 249–250). They also cover a similar subject earlier (ibid., pp. 189–194). Ryder (1996, pp. 382–384) also covers this particular derivation.

We will divide this into a couple parts to give it more structure.

Theorem 1.1 (Surface of unit sphere). *The surface of the d-dimensional unit sphere, i.e. the one where the surface itself has dimension $d-1$ is given by*

$$\int \mathrm{d}\Omega_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}.$$

You don't 'need' to... :)

Proof. The measure is given by Equation (3) on the problem set:

$$d^d p = P^{d-1} dP d\phi \sin\theta_1 d\theta_1 \sin^2\theta_2 d\theta_2 \dots \sin^{d-2}\theta_{d-1} d\theta_{d-2}.$$

We think that this is unnecessarily hard to read. We write it without the ellipsis and upright exterior derivative operators like so:

(...start from here, that is)

$$\mathrm{d}^d p = P^{d-1}\,\mathrm{d}P\,\mathrm{d}\phi \prod_{k=1}^{d-2} \sin(\theta_k)^k\,\mathrm{d}\theta_k$$

The unit sphere has $P = 1$. We will set P to unity. Then we can integrate over it and will end up with the area of the unit sphere.

$$\int \mathrm{d}\Omega_d = \int_0^{2\pi} \mathrm{d}\phi \prod_{k=1}^{d-2} \int_0^{\pi} \sin(\theta_k)^k\,\mathrm{d}\theta_k$$

The ϕ integral will just give 2π, but the other factors are not that easily computed. Mathematica can compute the θ_k integral and gives expressions with multiple Γ-functions.

Since we are not able to perform the integration of the factors which yield hypergeometric ${}_2F_1$ functions by hand, we will use the derivation by Peskin and Schroeder (1995, p. 249). They start by using the simplest Gaussian integral:

$$[\sqrt{\pi}]^d = \left[\int \mathrm{d}x\, \exp(-x^2)\right]^d$$

Then they just factor out the integral and chose $\mathbf{R}^d$ as the domain of integration. The exponential factors are grouped together such that there is a sum in the integration.

$$= \int \mathrm{d}^d x\, \exp\left(-\sum_{i=1}^{d} x_i^2\right)$$

Now they change into those generalized polar coordinates with $P = x$.

$$= \int x^{d-1}\,\mathrm{d}x\,\mathrm{d}\phi \prod_{k=1}^{d-2} \sin(\theta_k)^k\,\mathrm{d}\theta_k \exp\left(-\sum_{i=1}^{d} x_i^2\right)$$

Since we are in polar coordinates, we can apply the Pythagorean theorem d times in the argument of the exponential and write it as $-\boldsymbol{x}^2$. We also move the angular dependence to the front of the expression.

$$= \int \mathrm{d}\phi \prod_{k=1}^{d-2} \sin(\theta_k)^k\,\mathrm{d}\theta_k \int \mathrm{d}x\,x^{d-1} \exp\left(-x^2\right)$$

The first integral is just the surface area of the unit sphere. The second integral needs the substitution $z := x^2$ with $\mathrm{d}z = 2x\,\mathrm{d}x$ and therefore $\mathrm{d}x = \mathrm{d}z/[2x]$.

$$= \frac{1}{2} \int \mathrm{d}\Omega_d \int \mathrm{d}z\,z^{d/2-1} \exp(-z)$$

And this z integral is $\Gamma(d/2)$.

$$= \frac{1}{2}\Gamma\left(\frac{d}{2}\right) \int \mathrm{d}\Omega_d$$

We reorder the parts and obtain the equation we wanted to show. □

Now that we are done with the proof, we realized that Equation (5) from the problem set will help with the sine integrals. This is also the way Ryder (1996, p. 383) works with the generalized polar coordinates. We will leave it like it currently is.

Since we were not able to get to the correct result for the next thing, writing "Theorem" and "Proof" next to obviously wrong things looks rather stupid.

Theorem 1.2 (Radial integration). *The radial integral has the following solution:*

$$\int_0^\infty \mathrm{d}P \frac{P^{d-1}}{[P^2 - A]^n} = \frac{[-1]^n i}{2} \frac{\Gamma\left(n - \frac{d}{2}\right)\Gamma\left(\frac{d}{2}\right)}{\Gamma(n)} A^{\frac{d}{2}-n}.$$

✓

Proof. The trick is to find the right substitution; twice. Peskin and Schroeder (1995, p. 250) give them in their account, they just do this for the $n = 2$ case and have a plus sign in front of their A. First one has to substitute

$$\rho := P^2, \qquad \mathrm{d}\rho = 2P\,\mathrm{d}P.$$

Using that, we obtain

$$\int_0^\infty \mathrm{d}P \frac{P^{d-1}}{[P^2 - A]^n} = \int_0^\infty \mathrm{d}\rho \frac{\rho^{\frac{d}{2}-1}}{[\rho - A]^n}.$$

Then we have to substitute again. This time we use

$$z := \frac{A}{\rho - A}, \qquad \mathrm{d}z := -\frac{A}{[\rho - A]^2}\mathrm{d}\rho.$$

Using this, we can rewrite the integral again. The bounds are transformed like $0 \mapsto 1$ and $\infty \mapsto 0$. We exchange the bounds and also get rid of the minus sign in the substation in one step.

$$= \frac{1}{2}\int_0^1 \mathrm{d}z \frac{[\rho - A]^2}{A} \frac{\rho^{\frac{d}{2}-1}}{[\rho - A]^n}$$

Now we cancel the $[\rho - A]^2$. Then we still need to replace all the ρ by z.

$$= \frac{1}{2}\int_0^1 \mathrm{d}z \frac{1}{A} \frac{\rho^{\frac{d}{2}-1}}{[\rho - A]^{n-2}}$$

The denominator can be simplified like so:

$$\frac{1}{[\rho - A]^{n-2}} = \frac{A^{n-2}}{[\rho - A]^{n-2}} A^{2-n} = z^{n-2} A^{2-n}.$$

This simplifies the whole expression to

$$= \frac{1}{2}\int_0^1 \mathrm{d}z\, \rho^{\frac{d}{2}-1} A^{1-n} z^{n-2}.$$

Then we need to replace the left ρ with z. We have

$$z = \frac{A}{\rho - A} \iff \rho - A = \frac{A}{z} \iff \rho = A\left[\frac{1}{z} + 1\right].$$

Using that, we obtain

$$= \frac{1}{2}\int_0^1 \mathrm{d}z\, A^{\frac{d}{2}-n} \left[\frac{1}{z} + 1\right]^{\frac{d}{2}-1} z^{n-2}.$$

Then

$$\frac{1}{z}+1=\frac{1+z}{z}$$

allows us to cancel some of the z and we obtain some $[1+z]$.

$$=\frac{1}{2}\int_0^1 \mathrm{d}z\, A^{\frac{d}{2}-n} z^{n-1-\frac{d}{2}}[z+1]^{\frac{d}{2}-1}$$

At this point we can recognize the Beta function. We have

$$a=n-1-\frac{d}{2}, \qquad -2-a-b=\frac{d}{2}-1 \iff -a-b=\frac{d}{2}+1 \iff b=-n$$

We can then write the expression using the Beta function.

$$=\frac{1}{2}A^{\frac{d}{2}-n}B\left(n-\frac{d}{2},1-n\right)$$

And then we can expand the Beta function.

$$=\frac{1}{2}A^{\frac{d}{2}-n}\frac{\Gamma\left(n-\frac{d}{2}\right)\Gamma(1-n)}{\Gamma\left(1-\frac{d}{2}\right)}$$

Peskin and Schroeder (1995, p. 250) get usable Γ functions with their plus sign. We cannot do that since our denominator contains a minus sign. In a previous iteration of this problem we have mixed it up and had a minus sign in the denominator and a plus sign in the z substitution. With that we obtain

$$\frac{1}{2}A^{\frac{d}{2}-n}\frac{\Gamma\left(n-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}\right)}{\Gamma(n)}.$$

✓

This is the result Peskin and Schroeder (ibid., p. 250) have, it does not lead to the desired expression in Equation (2) on the problem set.

Maybe it is possible to smuggle an imaginary unit into there and get a better result. □

With those theorems at hand, we can attempt to prove Equation (2) from the problem set. We start by replacing the measure in the integral. $\boldsymbol{p}^2$ does not depend on the angles and we can replace that with P^2. The angular part from the measure is just the surface of the sphere which we know from Theorem 1.1.

$$\int \frac{\mathrm{d}^d p}{[2\pi]^d}\frac{1}{[\boldsymbol{p}^2-A]^n}=\frac{1}{[2\pi]^d}\frac{2\pi^{d/2}}{\Gamma(d/2)}\int_0^\infty \mathrm{d}P\frac{P^{d-1}}{[P^2-A]^n}$$

2.5/4

We can directly simplify the fraction.

$$=\frac{1}{2^{d-1}\pi^{d/2}\Gamma(d/2)}\int_0^\infty \mathrm{d}P\frac{P^{d-1}}{[P^2-A]^n}$$

Now we need some expression for this integral. Theorem 1.2 should give it us, but we were not able to prove it or figure out what it should actually be. We do not have it, so we cannot further simplify this.

1.3 Vertex rule

In the ϕ^4 theory we had so far, the interaction term was

$$\frac{\lambda}{4!}\phi^4.$$

0.5/0.5

Now we have

$$\frac{\lambda}{4!}\mu^{2[2-\omega]}\phi^4.$$

✓

Therefore the momentum space Feynman rule changes from $-i\lambda$ to $-i\lambda\mu^{2[2-\omega]}$.

1.4 Tadpole diagram

We iterate through the Feynman rules in momentum space.

- The incoming and outgoing line give us a factor of one. ✓ ✓
- The vertex gives $-i\lambda\mu^{2[2-\omega]}$ as derived in the previous part of this problem.
- The loop will give us one propagator. This is

  $$\frac{i}{p^2 - m^2}.$$ ✓

  We have left the $+i\epsilon$ out.
- We impose momentum conservation at each vertex. Let the incoming momentum be q. The loop momentum was already chosen to be p. From the total momentum conversation we know that the outgoing momentum has to be q again. Keeping the direction in mind we have ✓

  $$q + p - p - q = 0,$$

  which means that the momentum is conserved at the vertex if we do those choices. It also means that the momentum is conserved for any value of p which is probably the cause for the singularities.
- The momentum p is not determined, so we have to integrate over it. This gives another factor

  $$\int \frac{d^{2\omega}p}{[2\pi]^{2\omega}}.$$ ✓
- The symmetry factor of this diagram is two due to the loop. ✓

Together we have

$$\begin{aligned} \mathrm{i}\mathcal{M} &= -\frac{1}{2}\mathrm{i}\lambda\mu^{2[2-\omega]} \int \frac{\mathrm{d}^{2\omega}p}{[2\pi]^{2\omega}} \frac{\mathrm{i}}{p^2 - m^2} \\ &= \frac{1}{2}\lambda\mu^{2[2-\omega]} \int \frac{\mathrm{d}^{2\omega}p}{[2\pi]^{2\omega}} \frac{1}{p^2 - m^2}. \end{aligned}$$

Except for the sign, this matches the diagram given by Ryder (1996, (9.15)). We already tried and failed to derive the value of this integral. We can just insert Equation (2) from the problem set. Here $n = 1$, $d = 2\omega$ and $A = m^2$.

$$= \frac{1}{2}\lambda\mu^{2[2-\omega]} \frac{-\mathrm{i}}{[4\pi]^{\omega}} \frac{\Gamma(1-\omega)}{\Gamma(1)} m^{2\omega-1}$$

Since $\Gamma(1) = 1$ we can simplify further.

$$= -\frac{\mathrm{i}\lambda\mu^{2[2-\omega]}}{2[4\pi]^{\omega}} \Gamma(1-\omega) m^{2\omega-1}$$

We write the Γ-function such that we can expand it in the next step.

$$= -\frac{\mathrm{i}\lambda\mu^{2[2-\omega]}}{2[4\pi]^{\omega}} \Gamma(-[\omega-1]+\epsilon) m^{2\omega-1}$$

1.5/2

Now we can use Equation (9) from the problem set.

$$= \frac{[-1]^{\omega}\mathrm{i}\lambda\mu^{2[2-\omega]}}{2[4\pi]^{\omega}[\omega+1]!} \left[\frac{1}{\epsilon} + \psi(\omega) + \mathrm{O}(\epsilon)\right] m^{2\omega-1}$$

The problem statement asks to expand the Γ-function. We did that. Now we are supposed to expand the term $[\ldots]^{2-\omega}$ neglecting terms $\mathrm{O}(\omega-2)$ or higher. How is that "$\mathrm{O}(\omega-2)$" to be interpreted? The only factor that looks like $[\ldots]^{2-\omega}$ is the term with the mass μ^2. There is no power series at this point, so we cannot expand the sum and ignore terms higher than this.

The Digramma function ψ can be expanded in various ways, for instance the harmonic series with the Euler-Mascheroni constant. That would also explain why the constant Euler-Mascheroni γ is given in the problem statement although it does not appear at all in the formulas given there. The harmonic series would have factors which are just numbers up to $1/[\omega-1]$. This series expansion is finite and does not contain any powers of any dimensionful quantity.

1.5 Symmetry factor

We hope that we have already included the symmetry factor in the previous subsection.

1.6 Fish diagram

1.7 Further expansion

References

Penrose, Roger (2005). *Road to Reality*. 1. New York: Alfred A. Knopf. ISBN: 0-679-45443-8.

Peskin, Michael E. and Daniel V. Schroeder (1995). *An Introduction to Quantum Field Theory*. Westview Press. ISBN: 978-0-201-50397-5.

Ryder, Lewis H. (1996). *Quantum Field Theory*. Cambridge University Press. ISBN: 978-0-521-47814-4.