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This is a problem set (as turned in) for the module physics755.

This problem set is not reviewed by a tutor. This is just what I have turned in.

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[disclaimer]

physics755 – Quantum Field Theory

Problem Set 11

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Group Tuesday – Ripunjay Acharya

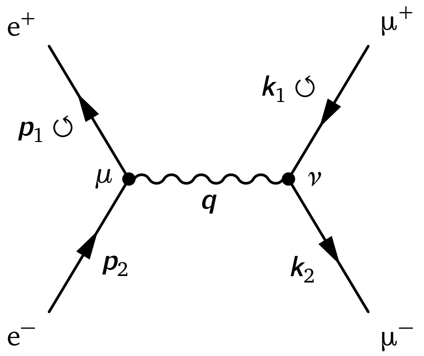
problem	achieved points	possible points
Muon pair production		8
Decay of the charged pion		7
total		15

This document consists of 11 pages.

1 Muon pair production

1.1 Invariant amplitude

The only contributing diagram is the following (time to the left):



The symbol "↺" says that the momentum is direction opposing to the particle number flow. This was easier than to figure out the direction of the actual fermion propagator in the FeynMP diagram and choosing an arrow from the symbols list or creating one with Ti*k*Z. Perhaps one day I will look into using Ti*k*Z for Feynman diagrams, but I do not want to spend the time on choosing the positions for the internal vertices by hand.

From this diagram we read off:

$$\mathrm{i}\mathcal{M} = [-\mathrm{i}e]^2 \bar{u}^{r'}(\boldsymbol{k}_1)\gamma^\mu v^r(\boldsymbol{k}_2)\frac{1}{[\boldsymbol{p}_1+\boldsymbol{p}_2]^2}\bar{v}^{s'}(\boldsymbol{p}_1)\gamma_\mu u^s(\boldsymbol{p}_2)$$

The denominator in the photon propagator can be replaced by the Mandelstam variable s. We go on to compute the appropriately averaged sum over squared amplitudes.

$$\frac{1}{4}\sum_{s,s',r,r'} |\mathrm{i}\mathcal{M}|^2$$
$$= \frac{1}{4}\frac{e^4}{s^2}\sum_{s,s',r,r'} \left[\bar{u}^{r'}(\boldsymbol{k}_1)\gamma^\mu v^r(\boldsymbol{k}_2)\bar{v}^{s'}(\boldsymbol{p}_1)\gamma_\mu u^s(\boldsymbol{p}_2)\right]^* \bar{u}^{r'}(\boldsymbol{k}_1)\gamma^\nu v^r(\boldsymbol{k}_2)\bar{v}^{s'}(\boldsymbol{p}_1)\gamma_\nu u^s(\boldsymbol{p}_2)$$

We use the first batch of relations from the previous exercise.

$$= \frac{1}{4}\frac{e^4}{s^2}\sum_{s,s',r,r'} \bar{v}^r(\boldsymbol{k}_2)\gamma^\mu u^{r'}(\boldsymbol{k}_1) \quad \bar{u}^s(\boldsymbol{p}_2)\gamma_\mu v^{s'}(\boldsymbol{p}_1) \quad \bar{u}^{r'}(\boldsymbol{k}_1)\gamma^\nu v^r(\boldsymbol{k}_2) \quad \bar{v}^{s'}(\boldsymbol{p}_1)\gamma_\nu u^s(\boldsymbol{p}_2)$$

We reorder the bilinears such that we can use the second batch of relations.

$$= \frac{1}{4}\frac{e^4}{s^2}\sum_{s,s',r,r'} \bar{v}^r(\boldsymbol{k}_2)\gamma^\mu u^{r'}(\boldsymbol{k}_1) \quad \bar{u}^{r'}(\boldsymbol{k}_1)\gamma^\nu v^r(\boldsymbol{k}_2) \quad \bar{v}^{s'}(\boldsymbol{p}_1)\gamma_\nu u^s(\boldsymbol{p}_2) \quad \bar{u}^s(\boldsymbol{p}_2)\gamma_\mu v^{s'}(\boldsymbol{p}_1)$$

Now we actually use the relations where m is the electron mass and M is the muon mass.

$$= \frac{1}{4}\frac{e^4}{s^2}\operatorname{tr}\left([\not{k}_1 + M]\gamma^\nu[\not{k}_2 - M]\gamma^\mu\right)\operatorname{tr}\left([\not{p}_1 - m]\gamma_\nu[\not{p}_2 + m]\gamma_\mu\right)$$

We neglect the mass of the electron. Since traces of an odd number of Dirac matrices vanish, there will not be any terms in first order of m. There is are only terms in second order of m, which is very small compared to M^2.

$$= \frac{1}{4}\frac{e^4}{s^2}\operatorname{tr}\left([\not{k}_1 + M]\gamma^\nu[\not{k}_2 - M]\gamma^\mu\right)\operatorname{tr}\left(\not{p}_1\gamma_\nu\not{p}_2\gamma_\mu\right)$$

We start with the last trace. We can extract the components of the momenta out of the trace.

$$= \frac{1}{4}\frac{e^4}{s^2}\operatorname{tr}\left([\not{k}_1 + M]\gamma^\nu[\not{k}_2 - M]\gamma^\mu\right)\boldsymbol{p}_1^\alpha \boldsymbol{p}_2^\beta \operatorname{tr}\left(\gamma_\nu\gamma^\alpha\gamma_\mu\gamma^\beta\right)$$

The trace of four Dirac matrices was computed on the last sheet. We just use this here.

$$= \frac{e^4}{s^2}\operatorname{tr}\left([\not{k}_1 + M]\gamma^\nu[\not{k}_2 - M]\gamma^\mu\right)p_1^\alpha p_2^\beta\left[\delta^\alpha_\nu\delta^\beta_\mu - \eta_{\nu\mu}\eta^{\alpha\beta} + \delta^\beta_\nu\delta^\alpha_\mu\right]$$

We perform the contractions over α and β.

$$= \frac{e^4}{s^2} \operatorname{tr}\left([\not{k}_1 + M]\gamma^\nu[\not{k}_2 - M]\gamma^\mu\right)\left[p_{1\mu}p_{2\nu} + p_{1\nu}p_{2\mu} - p_1^\alpha p_{2\alpha}\eta_{\mu\nu}\right]$$

Now we tend to the first trace. When we factor out the terms inside, we will get an order M^0 and an order M^2 term. The M^1 term will not contribute due to an odd number of Dirac matrices.

$$= \frac{e^4}{s^2}\left[\operatorname{tr}\left(\not{k}_1\gamma^\nu\not{k}_2\gamma^\mu\right) - M^2\operatorname{tr}(\gamma^\nu\gamma^\mu)\right]\left[p_{1\mu}p_{2\nu} + p_{1\nu}p_{2\mu} - p_1^\alpha p_{2\alpha}\eta_{\mu\nu}\right]$$

The trace has the same structure we had with $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ before. The second trace is just a multiple of the metric tensor.

$$= \frac{e^4}{s^2}\left[\left[k_1^\mu k_2^\nu + k_1^\nu k_2^\mu - k_1^\alpha k_{2\alpha}\eta^{\mu\nu}\right] - M^2\eta^{\mu\nu}\right]\left[p_{1\mu}p_{2\nu} + p_{1\nu}p_{2\mu} - p_1^\alpha p_{2\alpha}\eta_{\mu\nu}\right]$$

Then we can expand the left bracket and perform the contractions over μ and ν in the second term which we then get.

$$= \frac{e^4}{s^2}\left[k_1^\mu k_2^\nu + k_1^\nu k_2^\mu - k_1^\alpha k_{2\alpha}\eta^{\mu\nu}\right]\left[p_{1\mu}p_{2\nu} + p_{1\nu}p_{2\mu} - p_1^\alpha p_{2\alpha}\eta_{\mu\nu}\right] - \frac{M^2 e^4}{s^2}\left[2\boldsymbol{p}_1\cdot\boldsymbol{p}_2 - 4\boldsymbol{p}_1\cdot\boldsymbol{p}_2\right]$$

Now we look at the three terms that we will get when we expand the first bracket.

$$\begin{aligned} = \frac{e^4}{s^2}\Bigg[& k_1^\mu k_2^\nu\left[p_{1\mu}p_{2\nu} + p_{1\nu}p_{2\mu} - p_1^\alpha p_{2\alpha}\eta_{\mu\nu}\right] + k_1^\nu k_2^\mu\left[p_{1\mu}p_{2\nu} + p_{1\nu}p_{2\mu} - p_1^\alpha p_{2\alpha}\eta_{\mu\nu}\right] \\ & - k_1^\alpha k_{2\alpha}\eta^{\mu\nu}\left[p_{1\mu}p_{2\nu} + p_{1\nu}p_{2\mu} - p_1^\alpha p_{2\alpha}\eta_{\mu\nu}\right] - M^2\left[2\boldsymbol{p}_1\cdot\boldsymbol{p}_2 - 4\boldsymbol{p}_1\cdot\boldsymbol{p}_2\right]\Bigg] \end{aligned}$$

The third bracket can already be contracted.

$$\begin{aligned} = \frac{e^4}{s^2}\Bigg[& k_1^\mu k_2^\nu\left[p_{1\mu}p_{2\nu} + p_{1\nu}p_{2\mu} - \boldsymbol{p}_1\cdot\boldsymbol{p}_2\eta_{\mu\nu}\right] + k_1^\nu k_2^\mu\left[p_{1\mu}p_{2\nu} + p_{1\nu}p_{2\mu} - \boldsymbol{p}_1\cdot\boldsymbol{p}_2\eta_{\mu\nu}\right] \\ & + 2[\boldsymbol{k}_1\cdot\boldsymbol{k}_2][\boldsymbol{p}_1\cdot\boldsymbol{p}_2] + 2M^2\boldsymbol{p}_1\cdot\boldsymbol{p}_2\Bigg] \end{aligned}$$

The remaining terms which are not contracted are symmetric in their indices. We can combine those.

$$= 2\frac{e^4}{s^2}\left[k_1^\mu k_2^\nu\left[p_{1\mu}p_{2\nu} + p_{1\nu}p_{2\mu} - \boldsymbol{p}_1\cdot\boldsymbol{p}_2\eta_{\mu\nu}\right] + [\boldsymbol{k}_1\cdot\boldsymbol{k}_2][\boldsymbol{p}_1\cdot\boldsymbol{p}_2] + M^2\boldsymbol{p}_1\cdot\boldsymbol{p}_2\right]$$

And now we need to do the contractions.

$$= 2\frac{e^4}{s^2}\left[[\boldsymbol{k}_1\cdot\boldsymbol{p}_2][\boldsymbol{k}_2\cdot\boldsymbol{p}_1] + [\boldsymbol{k}_1\cdot\boldsymbol{p}_1][\boldsymbol{k}_2\cdot\boldsymbol{p}_2] - [\boldsymbol{k}_1\cdot\boldsymbol{k}_2][\boldsymbol{p}_1\cdot\boldsymbol{p}_2] + [\boldsymbol{k}_1\cdot\boldsymbol{k}_2][\boldsymbol{p}_1\cdot\boldsymbol{p}_2] + M^2\boldsymbol{p}_1\cdot\boldsymbol{p}_2\right]$$

The middle terms cancel, so we only have three terms left.

$$= 2\frac{e^4}{s^2}\left[[\boldsymbol{k}_1 \cdot \boldsymbol{p}_2][\boldsymbol{k}_2 \cdot \boldsymbol{p}_1] + [\boldsymbol{k}_1 \cdot \boldsymbol{p}_1][\boldsymbol{k}_2 \cdot \boldsymbol{p}_2] + M^2 \boldsymbol{p}_1 \cdot \boldsymbol{p}_2\right]$$

The actual result should be: (Peskin and Schroeder 1995, (5.10))

$$\frac{1}{4}\sum_{\text{spins}} |\mathscr{M}|^2 = \frac{8e^4}{q^4}\left[[\boldsymbol{p}_1 \cdot \boldsymbol{k}_1][\boldsymbol{p}_2 \cdot \boldsymbol{k}_2] + [\boldsymbol{p}_1 \cdot \boldsymbol{k}_2][\boldsymbol{p}_2 \cdot \boldsymbol{k}_1] + M^2 \boldsymbol{p}_1 \cdot \boldsymbol{p}_2\right].$$

So we are missing a factor of four.

1.2 Differential cross section

The differential cross section in the center of mass system should only depend on the scattering angle θ and the energy E each electron brings into the collision. We again use the center of mass diagram which we reproduced in Figure 1.

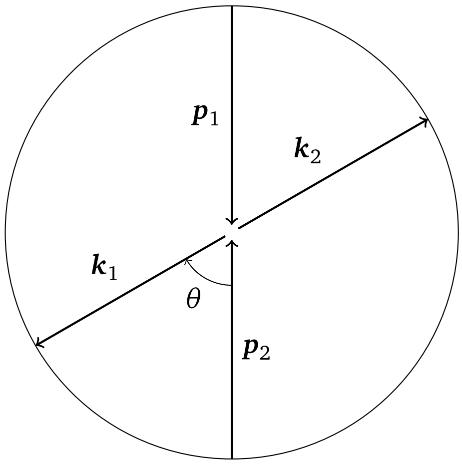


Figure 1: Scattering of two particles in the center of mass system. Shown are the momentum three-vectors. The gap in the center is just for presentation purposes.

From the diagram one can read off the relations between the three-momenta and compute the following scalar products of the four-momenta.

- The spatial parts $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ are the negatives of each other. The electrons are assumed to be massless, so the norm of the three-momenta is just the energy itself.

$$\boldsymbol{p}_1 \cdot \boldsymbol{p}_2 = E^2 - \boldsymbol{p}_1 \cdot \boldsymbol{p}_2 = 2E^2$$

- Mixed terms will include the scattering angle.

$$\boldsymbol{k}_1 \cdot \boldsymbol{p}_1 \overset{1}{=} E^2 - |\boldsymbol{k}_1||\boldsymbol{p}_1|\cos(\theta) \overset{2}{=} E^2 - E|\boldsymbol{k}_1|\cos(\theta) \overset{3}{=} E^2\left[1 - \frac{|\boldsymbol{k}|}{E}\cos(\theta)\right]$$

About the steps: (1) We have used the definition of the scalar product of four-vectors and the angle relation of the three-vector scalar product. (2) The norm of the incoming momentum is the energy since we assumed the electron to be massless. (3) We factor out E. The norms of $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ are the same, so we omit the index when we refer to the norm only.

- Now we look at the other mixed terms. The steps are the same, just that the angle is shifted by π.

$$\boldsymbol{k}_2 \cdot \boldsymbol{p}_1 = E^2 - |\boldsymbol{k}|\,|\boldsymbol{p}_1| \cos(\theta + \pi) = E^2 + E|\boldsymbol{k}|\cos(\theta) = E^2 \left[1 + \frac{|\boldsymbol{k}|}{E}\cos(\theta)\right]$$

We can take those terms to express the invariant matrix element in this coordinate system. We will use the version from Peskin and Schroeder (1995, (5.10)) to have a chance to get the correct answer.

$$\begin{aligned}\frac{1}{4}\sum_{\text{spins}} |\mathscr{M}|^2 &= \frac{8e^4}{q^4}\left[[\boldsymbol{p}_1 \cdot \boldsymbol{k}_1][\boldsymbol{p}_2 \cdot \boldsymbol{k}_2] + [\boldsymbol{p}_1 \cdot \boldsymbol{k}_2][\boldsymbol{p}_2 \cdot \boldsymbol{k}_1] + M^2 \boldsymbol{p}_1 \cdot \boldsymbol{p}_2\right] \\ &= \frac{8e^4}{16E^4}\left[E^4\left[1 - \frac{|\boldsymbol{k}|}{E}\cos(\theta)\right]^2 + E^4\left[1 + \frac{|\boldsymbol{k}|}{E}\cos(\theta)\right]^2 + 2E^2M^2\right]\end{aligned}$$

We expand the squares. The middle terms will just cancel due to the opposing signs.

$$\begin{aligned}&= \frac{e^4}{2E^4}\left[2E^4 + 2E^4\left[\frac{|\boldsymbol{k}|}{E}\cos(\theta)\right]^2 + 2E^2M^2\right] \\ &= e^4\left[1 + \frac{|\boldsymbol{k}|^2}{E^2}\cos(\theta)^2 + \frac{M^2}{E^2}\right]\end{aligned}$$

We know that $E^2 = |\boldsymbol{k}^2| + M^2$ and therefore we can write this as

$$= e^4\left[1 + \frac{E^2 - M^2}{E^2}\cos(\theta)^2 + \frac{M^2}{E^2}\right].$$

We expand the fraction.

$$= e^4\left[1 + \cos(\theta)^2 - \frac{M^2}{E^2}\cos(\theta)^2 + \frac{M^2}{E^2}\right]$$

Then we can factor out the cosine squared and obtain the result also given by Peskin and Schroeder (ibid., (5.11)).

$$= e^4\left[1 + \frac{M^2}{E^2} + \left[1 - \frac{M^2}{E^2}\right]\cos(\theta)^2\right]$$

From here we can plug this into Equation (2) given on the problem set and obtain the differential cross section.

$$\begin{aligned}\frac{\mathrm{d}\sigma_{\mathrm{cm}}}{\mathrm{d}\cos\theta} &= \frac{1}{2E_{\mathrm{cm}}^2}\frac{|\boldsymbol{k}|}{16\pi^2 E_{\mathrm{cm}}}\frac{1}{4}\sum_{\mathrm{spins}}|\mathcal{M}|^2 \\ &= \frac{1}{2E_{\mathrm{cm}}^2}\frac{|\boldsymbol{k}|}{16\pi^2 E_{\mathrm{cm}}}e^4\left[1+\frac{M^2}{E^2}+\left[1-\frac{M^2}{E^2}\right]\cos(\theta)^2\right]\end{aligned}$$

We can simplify the first term a bit.

$$= \frac{e^4}{32\pi^2 E_{\mathrm{cm}}^3}\sqrt{E^2-M^2}\left[1+\frac{M^2}{E^2}+\left[1-\frac{M^2}{E^2}\right]\cos(\theta)^2\right]$$

We extract one factor of the energy E out of the square root to make it look like the other factors in the large bracket.

$$= \frac{e^4}{32\pi^2 E_{\mathrm{cm}}^3}E\sqrt{1-\frac{M^2}{E^2}}\left[1+\frac{M^2}{E^2}+\left[1-\frac{M^2}{E^2}\right]\cos(\theta)^2\right]$$

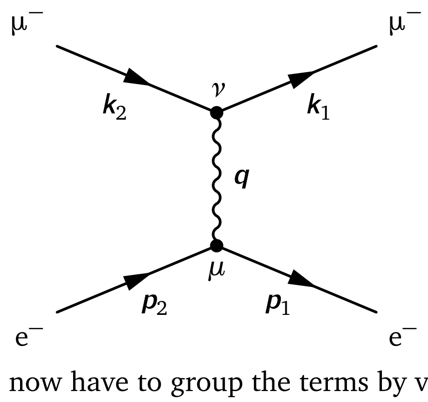
The relation between single particle energy E and the center of mass energy is $E_{\mathrm{cm}} = 2E$. The fine structure constant is $\alpha = e^2/[4\pi]$ We cancel the one factor of E, introduce the fine structure constant and obtain the final result.

$$= \frac{\alpha^2}{4E_{\mathrm{cm}}^2}\sqrt{1-\frac{M^2}{E^2}}\left[1+\frac{M^2}{E^2}+\left[1-\frac{M^2}{E^2}\right]\cos(\theta)^2\right]$$

This result matched the one given by Peskin and Schroeder (1995, (5.12)).

1.3 Electron muon scattering

The scattering process looks like this (time to the right):



We now have to group the terms by vertex again. Now it is not the creation and annihilation that make up a vertex but the electron and a muon coupling to the exchanged virtual photon. We will keep the assignment of the momenta such that the electrons are labeled with $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ and the muons with $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$. Now we have incoming particles with index 2 and outgoing particles with index 1.

To give some comparison, here are the invariant matrix elements for both processes:

$$\mathrm{i}\mathcal{M}_{\text{annihilation}} = [-\mathrm{i}e]^2 \, \bar{u}^{r'}(\boldsymbol{k}_1)\gamma^\mu v^r(\boldsymbol{k}_2)\frac{1}{[\boldsymbol{p}_1+\boldsymbol{p}_2]^2}\bar{v}^{s'}(\boldsymbol{p}_1)\gamma_\mu u^s(\boldsymbol{p}_2)$$
$$\mathrm{i}\mathcal{M}_{\text{scattering}} = [-\mathrm{i}e]^2 \, \bar{u}^{r'}(\boldsymbol{k}_1)\gamma_\mu u^r(\boldsymbol{k}_2)\frac{1}{[\boldsymbol{p}_1-\boldsymbol{p}_1]^2}\bar{u}^{s'}(\boldsymbol{p}_1)\gamma^\mu u^s(\boldsymbol{p}_2)$$

There is only u and no v since we only look at particles and there are no anti-particles around. The couplings are still the same, the electrons and the muons never coupled directly to each other. So each vertex only has one lepton flavor. Since the direction of the momentum have changed, the transfered momentum $\boldsymbol{q}$ is now the difference of the electron momenta.

This also changes the kinematics which are now shown in Figure 2. The scalar products need to be evaluated again as the masses changes in the respective terms. In the center of mass frame the three-momenta of the two incident particles sum to zero. This also holds true when their mass differs. The energy of the particles is not the same, however.

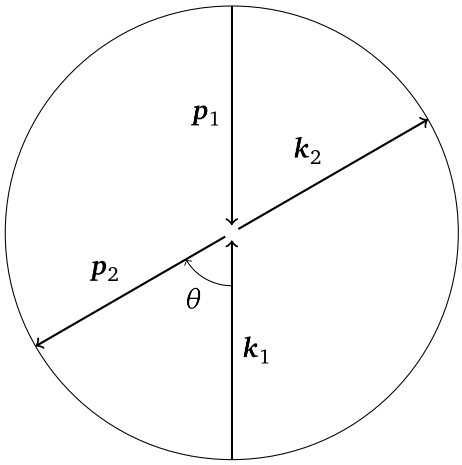


Figure 2: Same as Figure 2, just with exchanged particles.

- The electrons are massless, so the norm of the momentum is their energy.

  $$\boldsymbol{p}_1 \cdot \boldsymbol{p}_2 = E_\mathrm{e}^2 - |\boldsymbol{p}|^2\cos(\theta) = E_\mathrm{e}^2[1-\cos(\theta)]$$

- Similarly, we get the same result for the muon momenta.

  $$\boldsymbol{k}_1 \cdot \boldsymbol{k}_2 = E_\mu^2[1-\cos(\theta)]$$

- The mixed one now indirectly contains the mass M of the muon.

  $$\boldsymbol{k}_1 \cdot \boldsymbol{p}_1 = E_\mathrm{e}E_\mu - |\boldsymbol{k}_1|\,|\boldsymbol{p}_1|\cos(\pi) = E_\mathrm{e}E_\mu + E_\mathrm{e}|\boldsymbol{k}| = E_\mathrm{e}E_\mu\left[1+\frac{|\boldsymbol{k}|}{E_\mu}\right]$$

  In the limit $M \to 0$ and with that $|\boldsymbol{k}| \to E_\mu$ this will give the same $2E^2$ we had when we looked at the collision of two massless electrons.

- The mixed one now indirectly contains the mass M of the muon.

$$\boldsymbol{k}_1 \cdot \boldsymbol{p}_2 = E_\mathrm{e} E_\mu + |\boldsymbol{k}_1|\,|\boldsymbol{p}_2| \cos(\theta) = E_\mathrm{e} E_\mu + E_\mathrm{e} |\boldsymbol{k}| \cos(\theta) = E_\mathrm{e} E_\mu \left[1 + \frac{|\boldsymbol{k}|}{E_\mu} \cos(\theta) \right]$$

2 Decay of the charged pion

The diagram given on the problem set looks like time would go in the "←" direction. Every other time direction would mean that the arrows would go against the particle number flow.

2.1 Momentum space rule

The "diagram equals formula" expression on the problem set probably means that the vertex has the given form and we are supposed to arrive at this form.

The perturbation series is build with the Dyson series and the interacting part of the Hamiltonian. We only have the Lagrangian given, so we would need to derive the Hamiltonian first. The free Hamiltonian should be same as the Lagrangian, just with the derivatives of the fields replaced by the momenta.

$$H_\text{free} \rightsquigarrow \phi^*_{,\mu} \phi^{,\mu} - m_\pi^2 \phi^* \phi + \bar{\psi}[\mathrm{i}\not{\partial} - m_\mu]\psi + \bar{\chi}\mathrm{i}\not{\partial}\chi$$

To derive the interaction Hamiltonian we need to compute the canonical momenta and build up the Hamiltonian from that. The momentum for ϕ is

$$\pi^\phi_\mu = \phi^*_{,\mu} + \frac{G}{\sqrt{2}} f_\pi \left[\bar{\psi}\gamma^\mu [1 - \gamma^5] \chi\right].$$

When building the Hamiltonian with the pattern

$$H = \Upsilon^{,\mu} \pi^\Upsilon_\mu - L$$

where $\Upsilon \in \{\psi, \bar{\psi}, \phi, \phi^*, \chi, \bar{\chi}\}$, the interaction term beginning with G will just cancel. At first sight, there is no interaction in this theory, then.

However, the Hamiltonian lives in the cotangent fiber bundle instead of the tangent fiber bundle of the states. We need to express terms like $\phi^*_{,\mu}$ with the momenta and therefore get the interacting term back:

$$\phi^*_{,\mu} = \pi^\phi_\mu - \frac{G}{\sqrt{2}} f_\pi \left[\bar{\psi}\gamma^\mu [1 - \gamma^5] \chi\right].$$

The Hamiltonian contains

$$\phi^{,\mu} \pi^\phi_\mu + \text{h.c.}$$

where we need to replace $\phi^{,\mu}$. We have derived the expression for the complex conjugate, so we need to insert the complex conjugated of the above expression. The interacting term should be something along the lines of

$$\frac{G}{\sqrt{2}} f_\pi \pi_\mu^\phi \left[\bar{\psi} \gamma^\mu [1 - \gamma^5] \chi \right]^* + \text{h.c.}.$$

The states that we want to calculate matrix elements with look like

$$\langle \mu^-(\boldsymbol{k})\, \bar{\nu}_\mu(\boldsymbol{p}) | \dots | \pi^-(\boldsymbol{q}) \rangle .$$

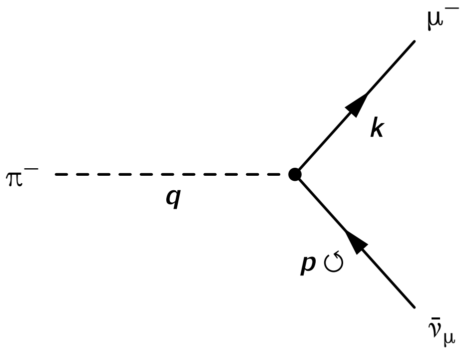
For momentum conservation we need to have $\boldsymbol{q} = \boldsymbol{k} + \boldsymbol{p}$.

The only contractions that are allowed are ψ with the muon, χ with the neutrino and ϕ with the pion. The ψ and $\bar{\psi}$ must only be contracted with incoming particles and antiparticles, respectively. It is the opposite way when looking at out-states. Since we have an outgoing leptons, we will need to have ψ and $\bar{\chi}$ in our interacting term that we can use in the matrix element.

We might be able to expand all the fields in terms of ladder operators and expand the states in terms of them as well. Then we would see which contractions are possible and also get the explicit expression for this matrix element. Assuming that the rules for the incoming edges are to be the same for fermions and bosons (that is $u(\boldsymbol{p})$ and so on for fermions and just 1 for bosons), we might be able to read off the expression for the vertex.

2.2 Amplitude for negative pion decay

We look at the following process:



Assuming that the vertex gives the factor given on the problem set and the other rules for fermions do not change, the invariant matrix element is

$$\mathrm{i}\mathcal{M} = \frac{G}{\sqrt{2}} f_\pi \bar{u}^{s'}(\boldsymbol{k}) [\not{p} + \not{k}] [\mathbf{1} - \gamma^5]\, v^s(\boldsymbol{p}).$$

Using this, we can write down the modulus squared and summed over all the spins. There is no factor to divide by since all the spin states are out-states.

$$|\mathcal{M}|^2 = \sum_{s,s'} \frac{G^2 f_\pi^2}{2} \left[\bar{u}^{s'}(\boldsymbol{k}) [\not{p} + \not{k}] [\mathbf{1} - \gamma^5]\, v^s(\boldsymbol{p}) \right]^* \bar{u}^{s'}(\boldsymbol{k}) [\not{p} + \not{k}] [\mathbf{1} - \gamma^5]\, v^s(\boldsymbol{p})$$

We use the relations from one of the previous sheets.

$$= \sum_{s,s'} \frac{G^2 f_\pi^2}{2} \bar{v}^s(\boldsymbol{p}) [\not{p} + \not{k}]^* [\mathbf{1} - \gamma^5] u^{s'}(\boldsymbol{k}) \bar{u}^{s'}(\boldsymbol{k}) [\not{p} + \not{k}] [\mathbf{1} - \gamma^5] v^s(\boldsymbol{p})$$

Now we can perform the spin sums where we already assume the neutrino to be massless.

$$= \frac{G^2 f_\pi^2}{2} \operatorname{tr}\left([\not{k} + m_\mu][\not{p} + \not{k}][\mathbf{1} - \gamma^5]\not{p}[\not{p} + \not{k}]^*[\mathbf{1} - \gamma^5]\right)$$

The term with the mass will not contribute since the trace would contain an odd number of Dirac matrices.

$$= \frac{G^2 f_\pi^2}{2} \operatorname{tr}\left(\not{k}[\not{p} + \not{k}][\mathbf{1} - \gamma^5]\not{p}[\not{p}^* + \not{k}^*][\mathbf{1} - \gamma^5]\right)$$

The fifth Dirac matrix anticommutes with every other one. So we should be able to anticommute it once …

$$= \frac{G^2 f_\pi^2}{2} \operatorname{tr}\left(\not{k}[\not{p} + \not{k}][\mathbf{1} - \gamma^5]\not{p}[\mathbf{1} + \gamma^5][\not{p}^* + \not{k}^*]\right)$$

… and a second time.

$$= \frac{G^2 f_\pi^2}{2} \operatorname{tr}\left(\not{k}[\not{p} + \not{k}][\mathbf{1} - \gamma^5][\mathbf{1} - \gamma^5]\not{p}[\not{p}^* + \not{k}^*]\right)$$

Then we use that the bracket is a projection operator, although the factor of 2 will be needed for that as well.

$$= G^2 f_\pi^2 \operatorname{tr}\left(\not{k}[\not{p} + \not{k}][\mathbf{1} - \gamma^5]\not{p}[\not{p}^* + \not{k}^*]\right)$$

We will move that to the very end such that we get closer to the trace identity with γ^5 given on the problem set in Equation (4).

$$= G^2 f_\pi^2 \operatorname{tr}\left(\not{p}[\not{p}^* + \not{k}^*]\not{k}[\not{p} + \not{k}][\mathbf{1} - \gamma^5]\right)$$

The complex conjugate of a Dirac matrix is not very nice. The application of the Hermitian conjugate will give a factor $\eta_{\mu\mu}$. For the transpose or complex conjugate alone there is not such a simple formula. We could save ourselves (or make it worse) by going back to the formula of complex conjugation of a bilinear and argue that the hermitian conjugate should have been used all the way. This would introduce this factor $\eta_{\mu\mu}$ which would also be a problem. So we are not really sure how to deal with this.

Other than that, we would look at the two terms we get from expanding the projection operator. The first term without γ^5 would vanish since it contains an odd number of Dirac matrices (except for the complex conjugated Dirac matrices, which might change the whole thing). Then we can tend to the remaining term and write $\not{p} = p_\mu \gamma^\mu$ and pull the component p_μ out of the trace since the trace is linear in its argument. Then we would be left with something where we can use the trace identity with.

2.3 Lifetime

Well, we cannot do this because we are missing the result from the previous problem. If we had, we would plug it into Equation (5) from the problem set. We would need to integrate the resulting expression over the whole solid angle

References

Peskin, Michael E. and Daniel V. Schroeder (1995). *An Introduction to Quantum Field Theory*. Westview Press. ISBN: 978-0-201-50397-5.