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[disclaimer]

physics755 – Quantum Field Theory

Problem Set 10

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problem	achieved points	possible points
Bhabha scattering		15
total		15

This document consists of 9 pages.

In the tutorial on 2015-06-23 we were a bit irritated that the momentum space Feynman rules given by Peskin and Schroeder (1995, p. 95) since the external legs had $\exp(-\mathrm{i}\boldsymbol{p} \cdot \boldsymbol{x})$ as an additional factor. We quickly checked another book and found that Tong (2007, p. 63) does not explicitly list the external legs, which means that he uses a factor of 1 there.

I think I can shed some light on this issue. The exact wording used is "For each external *point*" (emphasis mine). When we have such a point, we need a vector $\boldsymbol{x}$ in position space to get this point into the computation. The connection between momentum and space is given by $\exp(-\mathrm{i}\boldsymbol{p} \cdot \boldsymbol{x})$ then. Looking at (Peskin and Schroeder 1995, p. 115) where the rules for computing $\mathrm{i}\mathscr{M}$ are listed the wording is "For each external *line*" (emphasis mine) and a factor of 1. Three pages later, Peskin and Schroeder (ibid., p. 118) gives the Feynman rules for fermions, and there is no exponential factor either.

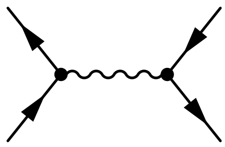
So I would suggest that the rules that we got from Peskin and Schroeder (ibid., p. 95) are not wrong in any sense. They are the wrong rules to use if we want to look at diagrams purely in the momentum state and not in the transition from position to momentum space like we did on the last homework problem set where we derived, or rather guessed, the rules for the momentum space. Taking away the external *points* will take away the exponentials.

1 Bhabha scattering

1.1 Contributing diagrams

Since we have two different kind of particles, they are distinguishable. This limits the number of diagrams that contribute to this process. We take the convention that the time goes to the right.

First diagram The first diagram has the annihilation of the particles which creates a real (because it is time-like) photon. Then this one creates a pair of electron of positron again:



The factors that we get for the parts of each diagram will be the same for the second diagram. However, the momentum conservation at the vertices will bind different momenta together. A different ordering of the terms might give a different overall sign.

The factors that we get for the first diagram are:

- The incoming fermion gives $u^s(\boldsymbol{p}_1)$.
- The incoming antifermion gives $\bar{v}^{s'}(\boldsymbol{p}_2)$.
- The outgoing fermion gives $\bar{u}^r(\boldsymbol{k}_1)$.
- The outgoing antifermion gives $v^{r'}(\boldsymbol{k}_2)$.
- The left vertex gives $-\mathrm{i}e\boldsymbol{\gamma}^\mu$.
- The right vertex gives $-\mathrm{i}e\boldsymbol{\gamma}^\nu$.
- The propagator gives $\mathrm{i}\eta_{\mu\nu}/\boldsymbol{q}^2$.

We need to impose momentum conservation at each vertex, this gives us $\boldsymbol{p}_1 + \boldsymbol{p}_2 = \boldsymbol{q}$ and $\boldsymbol{q} = \boldsymbol{k}_1 + \boldsymbol{k}_2$. There is no undetermined momentum left since we have a Feynman graph which is a tree (graph theory). Is it a Feynman tree, then?

Taking all those terms together we obtain

$$[-\mathrm{i}e]^2 \bar{v}^{s'}(\boldsymbol{p}_2)\boldsymbol{\gamma}^\mu u^s(\boldsymbol{p}_1)\frac{\mathrm{i}\eta_{\mu\nu}}{\boldsymbol{q}^2}\bar{u}^r(\boldsymbol{k}_1)\boldsymbol{\gamma}^\nu v^{r'}(\boldsymbol{k}_2).$$

To figure out the sign we look at the second order of the perturbation expansion. For QED, the interacting Hamiltonian density is $e\bar{\psi}\boldsymbol{\gamma}^\mu\psi A_\mu$. In second order we have this twice. So the core of our expression would be something along the lines of

$$\langle 0|\, b_{\boldsymbol{k}_2} a_{\boldsymbol{k}_1}\, \bar{\psi}(x)\boldsymbol{\gamma}^\mu\psi(x)A_\mu(x)\, \bar{\psi}(y)\boldsymbol{\gamma}^\nu A_\mu(y)\, a^\dagger_{\boldsymbol{p}_1} b^\dagger_{\boldsymbol{p}_2}\, |0\rangle\,.$$

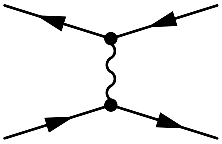
Our first diagram corresponds to the following contraction:

$$\langle 0|\, b_{\boldsymbol{k}_2} a_{\boldsymbol{k}_1}\, \bar{\psi}(x)\boldsymbol{\gamma}^\mu\psi(x)A_\mu(x)\, \bar{\psi}(y)\boldsymbol{\gamma}^\nu\psi(y)A_\mu(y)\, a^\dagger_{\boldsymbol{p}_1} b^\dagger_{\boldsymbol{p}_2}\, |0\rangle\,.$$

None of the contractions intersect, there are no further modifications needed to untangle this. This diagram has a no additional sign changes.

There is no symmetry factor to account for in this diagram.

Second diagram The second diagram is the more natural way of thinking about scattering. The two particles scatter by the exchange of a virtual photon:



We can take the same exact factors and build the invariant matrix element from this. The only difference is the position of the vertices. This only changes the internal momentum $\boldsymbol{q}$. Momentum conservation now gives us at the lower vertex

$$\boldsymbol{p}_2 = \boldsymbol{q} + \boldsymbol{k}_2.$$

The upper vertex will give us

$$\boldsymbol{p}_1 = -\boldsymbol{q} + \boldsymbol{k}_1.$$

So we have $\boldsymbol{q} = \boldsymbol{p}_2 - \boldsymbol{k}_2 = \boldsymbol{k}_1 - \boldsymbol{p}_1$.

We can directly go and look at the contractions.

$$\langle 0 | \, b_{\boldsymbol{k}_2} a_{\boldsymbol{k}_1} \, \bar{\psi}(x) \gamma^{\mu} \psi(x) A_{\mu}(x) \, \bar{\psi}(y) \gamma^{\nu} \psi(y) A_{\mu}(y) \, a^{\dagger}_{\boldsymbol{p}_1} b^{\dagger}_{\boldsymbol{p}_2} \, |0\rangle \,.$$

The $\bar{\psi}(\boldsymbol{y})$ in the middle of the expression needs to be moved in front of the $\psi(\boldsymbol{x})$ to untangle the lower contraction. This is one anticommutation, although we will get an additional term with $\delta^{(4)}(\boldsymbol{x} - \boldsymbol{y})$. After this exchange, we need to move the $\bar{\psi}(\boldsymbol{x})$ behind the $\psi(\boldsymbol{x})$. This now needs two anticommutations since we already moved the $\bar{\psi}(\boldsymbol{y})$ into there. This will give us no sign change but an additional term with $\delta^{(4)}(\mathbf{0})$ which looks a bit problematic.

Ignoring those extra commutators, we have the matrix element twice with different values for $\boldsymbol{q}$ and the second one with one minus sign with respect to the first one.

Both diagrams The matrix element then looks like this in full:

$$\mathrm{i}\mathcal{M} = -\mathrm{i}e^2 \eta_{\mu\nu} \left[\frac{1}{[\boldsymbol{p}_1 + \boldsymbol{p}_2]^2} - \frac{1}{[\boldsymbol{k}_1 - \boldsymbol{p}_1]^2} \right] \bar{v}^{s'}(\boldsymbol{p}_2) \gamma^{\mu} u^{s}(\boldsymbol{p}_1) \bar{u}^{r}(\boldsymbol{k}_1) \gamma^{\nu} v^{r'}(\boldsymbol{k}_2).$$

We can also use the mandelstam variables that will be introduced in the third part of this problem and write this more compact as:

$$\mathrm{i}\mathcal{M} = -\mathrm{i}e^2 \eta_{\mu\nu} \left[\frac{1}{s} - \frac{1}{t} \right] \bar{v}^{s'}(\boldsymbol{p}_2) \gamma^{\mu} u^{s}(\boldsymbol{p}_1) \bar{u}^{r}(\boldsymbol{k}_1) \gamma^{\nu} v^{r'}(\boldsymbol{k}_2).$$

The s in the index for the spin, whereas the s in the denominator is the Mandelstam variable.

1.2 Identities

First batch All the identities follow the same pattern. The first one is for different u and v, setting one to the other will give the second and third identity. We start by expanding the "bar".

$$[\bar{v}(\boldsymbol{p})\boldsymbol{\gamma}^\mu u(\boldsymbol{k})]^* = \left[v^\dagger(\boldsymbol{p})\boldsymbol{\gamma}^0\boldsymbol{\gamma}^\mu u(\boldsymbol{k})\right]^*$$

From the anticommutation of the two Dirac matrices, we get a factor $\eta_{\mu\mu}$.

$$= \eta_{\mu\mu}\left[v^\dagger(\boldsymbol{p})\boldsymbol{\gamma}^\mu\boldsymbol{\gamma}^0 u(\boldsymbol{k})\right]^*$$

Then we can absorb the $\boldsymbol{\gamma}^0$ into the u and give it a "bar".

$$= \eta_{\mu\mu}\left[v^\dagger(\boldsymbol{p})\boldsymbol{\gamma}^\mu\bar{u}^\dagger(\boldsymbol{k})\right]^*$$

The square bracket contains a scalar. For such, we can always add a transpose operation which does not change the value of the scalar. Together with the complex conjugate we will have a hermitian conjugation.

$$\begin{aligned}
&= \eta_{\mu\mu}\left[v^\dagger(\boldsymbol{p})\boldsymbol{\gamma}^\mu\bar{u}^\dagger(\boldsymbol{k})\right]^\dagger \\
&= \eta_{\mu\mu} u^\dagger(\boldsymbol{k})\boldsymbol{\gamma}^\mu\bar{v}(\boldsymbol{p})
\end{aligned}$$

This is almost the expression that we want. The Pauli matrices $\boldsymbol{\sigma}^\mu$ are hermitian. The Dirac matrices $\boldsymbol{\gamma}^\mu$ are build from those, except with an additional minus sign for $\mu \in \{1,2,3\}$. Those last three matrices are antihermitian, which will introduce a factor $\eta_{\mu\mu}$ again.

$$= [\eta_{\mu\mu}]^2 u^\dagger(\boldsymbol{k})\boldsymbol{\gamma}^\mu\bar{v}(\boldsymbol{p})$$

And since that factor can only be ± 1, it will be squared away. We arrive at the desired term.

$$= u^\dagger(\boldsymbol{k})\boldsymbol{\gamma}^\mu\bar{v}(\boldsymbol{p})$$

Second batch The wording sounds like we are allowed to just use the completeness relations without proof. So they are given by Peskin and Schroeder (1995, p. 49):

$$\sum_s u^s(\boldsymbol{p})\bar{u}^s(\boldsymbol{p}) = \not{\boldsymbol{p}} + m, \qquad \sum_s v^s(\boldsymbol{p})\bar{v}^s(\boldsymbol{p}) = \not{\boldsymbol{p}} - m.$$

In case there is only a blob of ink to see on the ink jet printout, that is a slashed four-vector p right after the equality sign.

We start with the first equation. We can move the summation over s in the middle and group the term for the next step.

$$\sum_{s,s'} \bar{u}^{s'}(\boldsymbol{p}')\boldsymbol{\gamma}^\mu u^s(\boldsymbol{p})\bar{u}^s(\boldsymbol{p})\boldsymbol{\gamma}^\nu u^{s'}(\boldsymbol{p}') = \sum_{s'} \bar{u}^{s'}(\boldsymbol{p}')\boldsymbol{\gamma}^\mu \left[\sum_s u^s(\boldsymbol{p})\bar{u}^s(\boldsymbol{p})\right]\boldsymbol{\gamma}^\nu u^{s'}(\boldsymbol{p}')$$

Here we can use the completeness relation and replace the contents of the bracket. We will explicitly write $m\mathbf{1}_4$ since adding a scalar to a matrix feels a bit imprecise.

$$= \sum_{s'} \bar{u}^{s'}(\boldsymbol{p}')\gamma^\mu \left[\not{\boldsymbol{p}} + m\mathbf{1}_4\right]\gamma^\nu u^{s'}(\boldsymbol{p}')$$

We add spinor indices (is that correct?) to the spinors such that we can move them around freely.

$$= \sum_{s'} \bar{u}^{s'}{}_a(\boldsymbol{p}')\left[\gamma^\mu \left[\not{\boldsymbol{p}} + m\mathbf{1}_4\right]\gamma^\nu\right]^a{}_b [u^{s'}]^b(\boldsymbol{p}')$$

The u can be moved to the very front of the equation. Everything that depends on s' is in the first terms, we add another bracket to isolate it for the next step.

$$= \left[\sum_{s'} [u^{s'}]^b(\boldsymbol{p}')\bar{u}^{s'}{}_a(\boldsymbol{p}')\right]\left[\gamma^\mu \left[\not{\boldsymbol{p}} + m\mathbf{1}_4\right]\gamma^\nu\right]^a{}_b$$

We think that we can still apply the completeness relation since we sum over s' first and then over the spinor indices a and b.

$$= \left[\not{\boldsymbol{p}}' + m\mathbf{1}_4\right]^b{}_a \left[\gamma^\mu \left[\not{\boldsymbol{p}} + m\mathbf{1}_4\right]\gamma^\nu\right]^a{}_b$$

Then the summation over both indices will result in a matrix multiplication (b) and a trace (a).

$$= \operatorname{tr}\left(\left[\not{\boldsymbol{p}}' + m\mathbf{1}_4\right]\gamma^\mu \left[\not{\boldsymbol{p}} + m\mathbf{1}_4\right]\gamma^\nu\right)$$

The second identity has a minus sign in the $\not{\boldsymbol{p}}'$ term since there are v spinors outside. The third one has the minus sign with the $\not{\boldsymbol{p}}$ term since the inner spinors are v instead of u. And in the last one there are only v, so there are minus signs in both spots.

Third batch This batch contains of assorted identities, we will just iterate over them.

1. The identity $\operatorname{tr}(\mathbf{1}) = 4$ is a bit weird. Either we assume that $\mathbf{1}$ is a 4×4 matrix and this is rather trivial, or there is something very deep to it. The identity matrix is the one which only has ones on the diagonal. Therefore the trace is the dimension of that matrix.

2. The Dirac matrices can be written in the chiral representation like this:

$$\gamma^\mu = \begin{pmatrix} 0 & \boldsymbol{\sigma}^\mu \\ \bar{\boldsymbol{\sigma}}^\nu & 0 \end{pmatrix}.$$

   Forming the product of two such Dirac matrices will yield a block diagonal matrix:

$$\gamma^\mu\gamma^\nu = \begin{pmatrix} \boldsymbol{\sigma}^\mu\bar{\boldsymbol{\sigma}}^\nu & 0 \\ 0 & \bar{\boldsymbol{\sigma}}^\mu\boldsymbol{\sigma}^\nu \end{pmatrix}.$$

   Any even number of Dirac matrices will give a block diagonal matrix. Adding one more Dirac matrix will make it an off-diagonal matrix which has no trace. Therefore

$$\operatorname{tr}\left(\prod_{i=1}^{2n+1} \gamma^{\mu_i}\right) = 0 \qquad \text{with } n \in \mathbf{N}.$$

3. We start with the anticommutation relation of the Dirac matrices. This gives us

$$\operatorname{tr}(\gamma^\mu \gamma^\nu) = \operatorname{tr}(-\gamma^\nu \gamma^\mu + 2\eta^{\mu\nu} \mathbf{1}_4) = -\operatorname{tr}(\gamma^\nu \gamma^\mu) + 8\eta^{\mu\nu},$$

where we have used the linearity and the first identity in the last step. The trace is cyclic, so we also know that

$$\operatorname{tr}(\gamma^\mu \gamma^\nu) = \operatorname{tr}(\gamma^\nu \gamma^\mu).$$

We can now rewrite the first equation as

$$2 \operatorname{tr}(\gamma^\mu \gamma^\nu) = 8\eta^{\mu\nu},$$

which will give the desired relation after dividing by a factor of two:

$$\operatorname{tr}(\gamma^\mu \gamma^\nu) = 4\eta^{\mu\nu}.$$

4. We will group the matrices again in pairs and use the relation derived for the second identity.

$$\operatorname{tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) = \operatorname{tr}\left(\operatorname{diag}\left(\boldsymbol{\sigma}^\mu \bar{\boldsymbol{\sigma}}^\nu, \bar{\boldsymbol{\sigma}}^\mu \boldsymbol{\sigma}^\nu\right) \operatorname{diag}\left(\boldsymbol{\sigma}^\rho \bar{\boldsymbol{\sigma}}^\sigma, \bar{\boldsymbol{\sigma}}^\rho \boldsymbol{\sigma}^\sigma\right)\right)$$

The product of the two block-diagonal matrices gives another block-diagonal matrix.

$$= \operatorname{tr}\left(\operatorname{diag}\left(\boldsymbol{\sigma}^\mu \bar{\boldsymbol{\sigma}}^\nu \boldsymbol{\sigma}^\rho \bar{\boldsymbol{\sigma}}^\sigma, \bar{\boldsymbol{\sigma}}^\mu \boldsymbol{\sigma}^\nu \bar{\boldsymbol{\sigma}}^\rho \boldsymbol{\sigma}^\sigma\right)\right)$$

The trace is the sum of the traces of this block-diagonal matrix.

$$= \operatorname{tr}\left(\boldsymbol{\sigma}^\mu \bar{\boldsymbol{\sigma}}^\nu \boldsymbol{\sigma}^\rho \bar{\boldsymbol{\sigma}}^\sigma\right) + \operatorname{tr}\left(\bar{\boldsymbol{\sigma}}^\mu \boldsymbol{\sigma}^\nu \bar{\boldsymbol{\sigma}}^\rho \boldsymbol{\sigma}^\sigma\right)$$

The only Pauli matrix that has a trace is the $\boldsymbol{\sigma}^0$ matrix. All other ones do not have a trace. This also means that they can be used as the generators of a special matrix group. If we square the Pauli matrices, we obtain a matrix that is proportional to the identity matrix. The "bar" will give a minus sign for the spatial indices, so we need to take that into account. If we have μ, ν, ρ and σ form two pairs, we only have matrices which are proportional to the identity. The trace will be ± 4 then. A metric tensor will allow us to cope with the minus signs by the "bar". So we have three possibilities to "contract" the four matrices. Looking at the signs that we get, we can write the result as

$$= 4\left[\eta^{\mu\nu}\eta^{\rho\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho} - \eta^{\mu\rho}\eta^{\nu\sigma}\right].$$

There are no other possibilities to get a non-zero trace. The product of two non-equal Pauli matrices is a third one which will have no trace. So if only two indices match up, we have a diagonal matrix times a traceless matrix which will be traceless. If no index matches up, the product of the four factors will be a single spatial Pauli matrix which has no trace either.

5. The $\operatorname{tr}(\boldsymbol{\gamma}^5)$ is zero because the matrix is given by

   $$\boldsymbol{\gamma}^5 = \operatorname{diag}(1,1,-1,-1)$$

   in the representation that we usually use.

6. If $\mu = \nu$ then the product $[\boldsymbol{\gamma}^\mu]^2$ will be $\eta_{\mu\mu}\mathbf{1}_4$. Together with $\boldsymbol{\gamma}^5$ this will be a traceless matrix. In the other case that $\mu \neq \nu$ the product will be an off-diagonal matrix, as shown for the second identity. The multiplication with $\boldsymbol{\gamma}^5$ will change the signs, but the main diagonal will not be populated. The matrix does not change a trace either.

1.3 Simplification with Mandelstam variables

We take the matrix element that we have computed in the first part of this problem and take the square of it.

$$\frac{1}{4}\sum_{s,s',r,r'} |\mathcal{M}|^2 = \frac{e^4}{4}\sum_{s,s',r,r'} \eta_{\mu\nu}\eta_{\rho\lambda}\left[\frac{1}{s}-\frac{1}{t}\right]^2 \\ \left[\bar{v}^{s'}(\boldsymbol{p}_2)\boldsymbol{\gamma}^\mu u^s(\boldsymbol{p}_1)\bar{u}^r(\boldsymbol{k}_1)\boldsymbol{\gamma}^\nu v^{r'}(\boldsymbol{k}_2)\right]^* \bar{v}^{s'}(\boldsymbol{p}_2)\boldsymbol{\gamma}^\rho u^s(\boldsymbol{p}_1)\bar{u}^r(\boldsymbol{k}_1)\boldsymbol{\gamma}^\lambda v^{r'}(\boldsymbol{k}_2)$$

We can use the relations from the first batch to remove the complex conjugation.

$$= \frac{e^4}{4}\sum_{s,s',r,r'} \eta_{\mu\nu}\eta_{\rho\lambda}\left[\frac{1}{s}-\frac{1}{t}\right]^2 \\ \bar{u}^s(\boldsymbol{p}_1)\boldsymbol{\gamma}^\mu v^{s'}(\boldsymbol{p}_2)\bar{v}^{r'}(\boldsymbol{k}_2)\boldsymbol{\gamma}^\nu u^r(\boldsymbol{k}_1)\bar{v}^{s'}(\boldsymbol{p}_2)\boldsymbol{\gamma}^\rho u^s(\boldsymbol{p}_1)\bar{u}^r(\boldsymbol{k}_1)\boldsymbol{\gamma}^\lambda v^{r'}(\boldsymbol{k}_2)$$

We need to reorder the three Dirac bilinears in order to use relations we showed in the second batch. We also use the metric tensors to lower the indices.

$$= \frac{e^4}{4}\sum_{s,s',r,r'} \left[\frac{1}{s}-\frac{1}{t}\right]^2 \\ \bar{u}^s(\boldsymbol{p}_1)\boldsymbol{\gamma}^\mu v^{s'}(\boldsymbol{p}_2)\underbrace{\bar{v}^{r'}(\boldsymbol{k}_2)\boldsymbol{\gamma}_\mu u^r(\boldsymbol{k}_1)\bar{u}^r(\boldsymbol{k}_1)\boldsymbol{\gamma}_\rho v^{r'}(\boldsymbol{k}_2)}_{\operatorname{tr}\left([\not{k}_2-m]\boldsymbol{\gamma}_\mu[\not{k}_1-m]\boldsymbol{\gamma}_\rho\right)}\bar{v}^{s'}(\boldsymbol{p}_2)\boldsymbol{\gamma}^\rho u^s(\boldsymbol{p}_1)$$

We have used the identity from the second batch for the middle terms. The two Dirac bilinears that are left can be converted into such a trace as well. Now all the spins are summed over and we do not have any summation signs left.

$$= \frac{e^4}{4}\left[\frac{1}{s}-\frac{1}{t}\right]^2 \operatorname{tr}\left([\not{k}_2-m]\boldsymbol{\gamma}_\mu[\not{k}_1-m]\boldsymbol{\gamma}_\rho\right)\operatorname{tr}\left([\not{p}_1+m]\boldsymbol{\gamma}^\mu[\not{p}_2-m]\boldsymbol{\gamma}^\rho\right)$$

Here we have a product of two traces instead of one big trace as given on the problem set. Since the product of traces does not have a direct nice identity, we did not know how to advance from here.

1.4 Massless limit

The momentum three-vectors in the center of mass frame are shown in Figure 1. All three-momenta lie on a circle since the sum of $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ has to be zero and this momentum is conserved in the scattering process. Since we are looking at elastic scattering, the energy of all particles involved is the same, since the rest mass is the same as well.

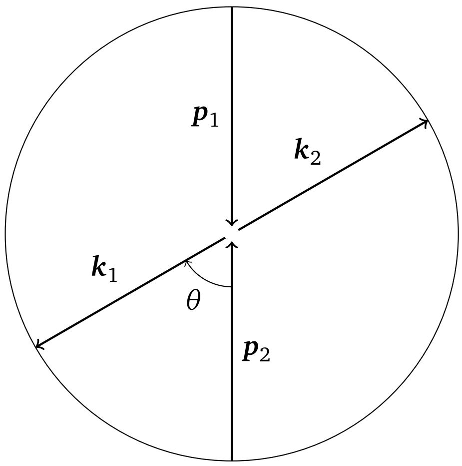


Figure 1: Scattering of two particles in the center of mass system. Shown are the momentum three-vectors. The gap in the center is just for presentation purposes.

We will look at the Mandelstam variables and write them out in the high energy limit. The variable s will just contain the energy. Note the change from four-vectors to three-vectors, which is hard to see with the ISO 80000-2 notation.

$$s = \boldsymbol{p}_1 \cdot \boldsymbol{p}_2 = E_{\text{cm}}^2 - \boldsymbol{p}_1 \cdot \boldsymbol{p}_2 = E_{\text{cm}}^2 + \boldsymbol{p}_1 \cdot \boldsymbol{p}_1 = 2E_{\text{cm}}^2$$

The variable t will contain a dependence on the angle.

$$t = -\boldsymbol{p}_1 \cdot \boldsymbol{k}_1 = -E_{\text{cm}}^2 + \boldsymbol{p}_1 \cdot \boldsymbol{k}_1 = -E_{\text{cm}}^2 + |\boldsymbol{p}_1||\boldsymbol{k}_1|\cos(\theta) = E_{\text{cm}}^2 [\cos(\theta) - 1]$$

And similarly we obtain an expression for u:

$$u = -\boldsymbol{p}_1 \cdot \boldsymbol{k}_2 = -E_{\text{cm}}^2 + \boldsymbol{p}_1 \cdot \boldsymbol{k}_2 = -E_{\text{cm}}^2 - \boldsymbol{p}_1 \cdot \boldsymbol{k}_1 = -E_{\text{cm}}^2 - E_{\text{cm}}^2 \cos(\theta) = -E_{\text{cm}}^2 [\cos(\theta) + 1].$$

The three summands from Equation (7) from the problem set can be expanded now with the new expressions for the Mandelstam variables. We start with Equation (8) from the problem set.

$$\frac{\mathrm{d}\sigma_{\text{cm}}}{\mathrm{d}\cos(\theta)} = \frac{|\mathscr{M}|^2}{32\pi E_{\text{cm}}^2}$$

Then we insert Equation (7). It only makes sense if the sum over all the spin states is already executed.

$$= \frac{1}{4\pi E_{\text{cm}}^2} \left[\frac{s^2 + u^2}{t^2} + \frac{t^2 + u^2}{s^2} + \frac{2u^2}{st} \right]$$

Now we use the expressions that we computed in the beginning of this part. We have already cancelled the terms with E_{cm}^4.

$$= \frac{1}{4\pi E_{\mathrm{cm}}^2}\left[\frac{4+[\cos(\theta)+1]^2}{[\cos(\theta)-1]^2}+\frac{[\cos(\theta)-1]^2+[\cos(\theta)+1]^2}{4}+\frac{[\cos(\theta)+1]^2}{[\cos(\theta)-1]^2}\right]$$

We used Mathematica to simplify this expression and arrived at the rather simple form

$$= \frac{[3+\cos(\theta)]^2}{8\pi E_{\mathrm{cm}}^2[\cos(\theta)-1]^2}.$$

Instead of the laborious steps you get an awesome looking graph of the differential cross section in Figure 2.

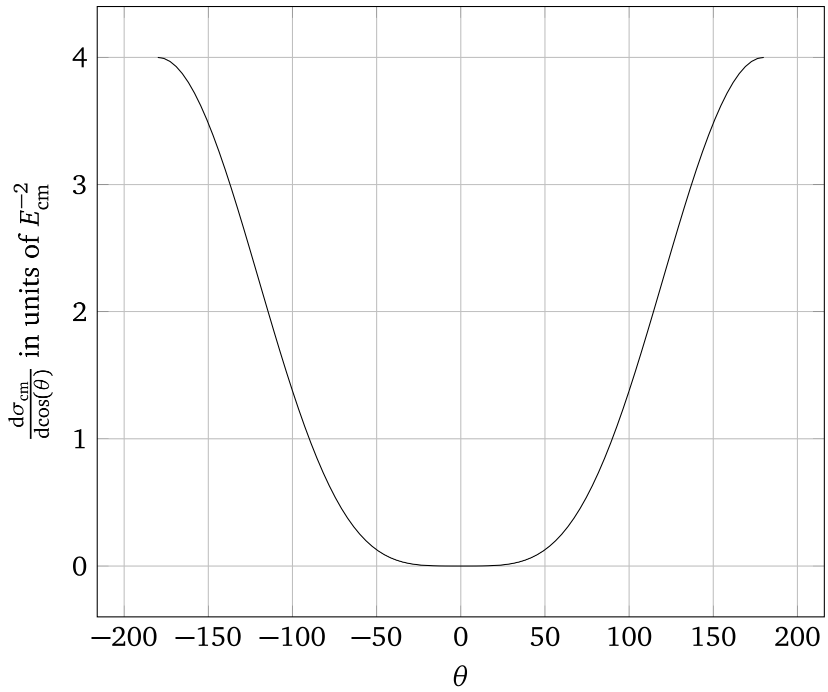


Figure 2: Differential cross section.

References

Peskin, Michael E. and Daniel V. Schroeder (1995). *An Introduction to Quantum Field Theory*. Westview Press. ISBN: 978-0-201-50397-5.

Tong, Daving (2007). *Quantum Field Theory*. URL: `http://www.damtp.cam.ac.uk/user/tong/qft.html`.