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physics755 – Quantum Field Theory

Problem Set 9

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problem	achieved points	possible points
The S-Matrix		15
total		15

This document consists of 8 pages.

1 The S-Matrix

1.1 Momentum conservation

This problem covers the material also shown by Peskin and Schroeder (1995, p. 94).

Momentum conservation at vertex Each vertex gives us a factor $-\mathrm{i}\lambda \int \mathrm{d}^4 z$, where z is the four-vector for the particular vertex. Each vertex has to have its own internal integration variable. Then each propagator gives us $D(x-y)$. Looking at the vertex and all the propagators (labeled with i) that end in this vertex, we get the following term, which we label A to continue working on it.

$$A := -\mathrm{i}\lambda \int \mathrm{d}^4 z \prod_i D(x_i - z)$$

To say anything about the momentum, we have to Fourier transform the propagator. That is given by Peskin and Schroeder (ibid., (4.46) as

$$D(x-y) = \int \frac{\mathrm{d}^4 p}{[2\pi]^4} \frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\epsilon} \exp(-\mathrm{i} p \cdot [x-y]) .$$

We insert this into A.

$$A = -\mathrm{i}\lambda \int \mathrm{d}^4 z \prod_i \int \frac{\mathrm{d}^4 p_i}{[2\pi]^4} \frac{\mathrm{i}}{\boldsymbol{p}_i^2 - m^2 + \mathrm{i}\epsilon} \exp\left(-\mathrm{i}\boldsymbol{p}_i \cdot [\boldsymbol{x}_i - \boldsymbol{z}]\right)$$

We can move all the z dependence to the very back of the expression to isolate the part which is of interest here.

$$= -\mathrm{i}\lambda \prod_i \int \frac{\mathrm{d}^4 p_i}{[2\pi]^4} \frac{\mathrm{i}}{\boldsymbol{p}_i^2 - m^2 + \mathrm{i}\epsilon} \exp(-\mathrm{i}\boldsymbol{p}_i \cdot \boldsymbol{x}_i) \int \mathrm{d}^4 z \exp(\mathrm{i}\boldsymbol{p}_i \cdot \boldsymbol{z})$$

We execute the product for the last term to get the expression that we can work with.

$$= -\mathrm{i}\lambda \left[\prod_i \int \frac{\mathrm{d}^4 p_i}{[2\pi]^4} \frac{\mathrm{i}}{\boldsymbol{p}_i^2 - m^2 + \mathrm{i}\epsilon} \exp(-\mathrm{i}\boldsymbol{p}_i \cdot \boldsymbol{x}_i)\right] \int \mathrm{d}^4 z \exp\left(\mathrm{i}\left[\sum_i \boldsymbol{p}_i\right] \cdot \boldsymbol{z}\right)$$

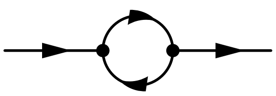
This last integral will give us the desired result of a momentum conserving δ-distribution.

$$= -\mathrm{i}\lambda \left[\prod_i \int \frac{\mathrm{d}^4 p_i}{[2\pi]^4} \frac{\mathrm{i}}{\boldsymbol{p}_i^2 - m^2 + \mathrm{i}\epsilon} \exp(-\mathrm{i}\boldsymbol{p}_i \cdot \boldsymbol{x}_i)\right] [2\pi]^4 \delta^{(4)}\left(\sum_i \boldsymbol{p}_i\right)$$

Each external line has a factor $\exp(-\mathrm{i}\boldsymbol{p} \cdot \boldsymbol{x})$ associated with it. So this works for external lines just the same as it does for propagators ending on the vertex.

Overall momentum conservation We have momentum conservation in each vertex as derived above. Just like with Kirchoff's law for the electric current (which follows from the continuity equation and that $\dot{\varrho} = 0$ in regular wires), momentum can be transfered between the vertices but is conserved in total.

Fixed internal momenta Each propagator brings a momentum with it. If there are too few propagators in the diagram, everything is fixed. Whenever there is a loop in the diagram, there is a freedom in momentum there, both can be increased by the same amount and total momentum is conserved.



Internal momenta are fixed if there is no closed path that connects vertices. So if the graph is a tree, all internal momenta are fixed by momentum conversion.

1.2 Momentum space Feynman rules

This problem covers the material also shown by Peskin and Schroeder (1995, p. 95).

The number of vertices should still give the order in λ, so we will associate the factor of $-\mathrm{i}\lambda$ with the vertices.

We have seen in the A that the remainder of the propagator is

$$\int \frac{\mathrm{d}^4 p}{[2\pi]^4} \frac{\mathrm{i}}{\boldsymbol{p}^2 - m^2 + \mathrm{i}\epsilon} \exp(-\mathrm{i}\boldsymbol{p} \cdot \boldsymbol{x})$$

It will be convenient to split this up into several parts. The integration over $\boldsymbol{p}$ becomes a rule that all internal momenta need to be integrated over in the end. Each propagator gets the fraction assigned. The exponential can be assigned to the external point connecting the propagator.

This way we can have the same rules that are given by Peskin and Schroeder (1995, p. 95):

1. Each propagator gets a factor

   $$\frac{\mathrm{i}}{\boldsymbol{p}^2 - m^2 + \mathrm{i}\epsilon}$$

   assigned.
2. Each vertex gets just the factor $-\mathrm{i}\lambda$, which then also determines the order of the diagram.
3. Each external point gets the exponential $\exp(-\mathrm{i}\boldsymbol{p} \cdot \boldsymbol{x})$. This reference to a space-time point is a problem since we do not want them here. In a momentum space prescription, there should not bet external points to begin with. There should only be incoming propagators.
4. Since we derived that total momentum at the vertices is conserved, we have to apply the derived δ-distribution and assert momentum conservation at each of the vertices.
5. All internal momenta have to integrated over.
6. And as before, we need to divide by the symmetry factor.

1.3 Case for Dirac and electromagnetic field

This problem covers the material also shown by Peskin and Schroeder (ibid., §4.7f).

Looking at Equation (7) from the problem set we see that the factor $\exp(-\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{x})$ comes from the application of a positive frequency operator ϕ_I^+ onto a free momentum state $|\boldsymbol{k}\rangle$. In the previous subproblem we looked at the Feynman rules for the scalar fields.

Fermions The fermions fulfil the Dirac equation and therefore need to be modeled with a four-component spinor instead of a single scalar field. Also, the fermionic fields anticommute which needs to be taken into account as well in the contractions. A sensible way to do this is using a generalized from of Wick's theorem where the time and normal ordering operators account for the sign changes.

The Dirac field also has distinct creation and annihilation operators, whereas the real scalar field only had particles which are their own antiparticles. So far we only had the real scalar field ϕ which occurred in even powers in the Hamiltonian. Now we have the fields $\bar\psi$ and ψ which occur in pairs to give bilinears with distinct transformation properties under the Lorentz group. The only nonzero contraction we can build is from a mixed pair of those, contractions between ψs or $\bar{\psi}$s respectively give a zero propagator.

The "external leg rule" then changes by the way the Dirac fields act on the pure momentum states:

$$\overline{\psi_{\mathrm{I}}(x)\,|\boldsymbol{k},s}\rangle_0 = \psi_{\mathrm{I}}(x)\sqrt{2E_{\boldsymbol{k}}}a_{\boldsymbol{k}}^{s\dagger}\,|0\rangle\,.$$

We expand the field ψ in term of positive and negative frequency operators. Only the positive frequency part is interesting since we evaluate a matrix element with single-momentum states on both sides eventually. Peskin and Schroeder (1995, (4.114))

$$= \int \frac{\mathrm{d}^3k'}{[2\pi]^3}\frac{1}{\sqrt{E_{\boldsymbol{k}'}}}\sum_{s'} a_{\boldsymbol{k}'}^{s'} u^{s'}(\boldsymbol{k}')\exp(-\mathrm{i}\boldsymbol{k}'\cdot x)\sqrt{2E_{\boldsymbol{k}}}a_{\boldsymbol{k}}^{s\dagger}\,|0\rangle\,.$$

The anticommutator of the ladder operators will give us a δ-distribution and we can simplify this to

$$= u^s(\boldsymbol{k})\exp(-\mathrm{i}\boldsymbol{k}\cdot x)\,|0\rangle\,.$$

Now we have the exact same exponential as there is in Equation (8) on the problem set. There is an additional upper spinor component $u^s(\boldsymbol{k})$. From this we would conclude that the factor for each external line would be $u^s(\boldsymbol{k})\exp(-\mathrm{i}\boldsymbol{k}\cdot x)$. Peskin and Schroeder (ibid., p. 118) state the "external leg rule" for fermions as just having $u^s(\boldsymbol{k})$, though. So the exponential is moved somewhere else?

The same argument can probably be made for the $\bar{\psi}$ as well, and then one would have $\bar{\psi}$ acting on antifermions, which will give a factor $\bar{v}^s(\boldsymbol{k})$.

In both cases, the action of the "hermitian conjugate and $\boldsymbol{\gamma}^0$ version also gets such a bar over the factor in the diagram. So reversal of the particle number flow adds or removes the bar over the spinor part.

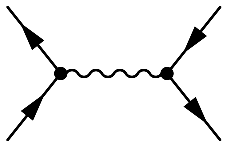
Photons As mentioned in front of part 6 of this exercise on the problem set, the interaction term is given by $H_{\mathrm{int}} = \mathrm{i}e\bar{\psi}\boldsymbol{\gamma}^\mu\psi A_\mu$. The "external leg rule" is given on the problem set as $-\mathrm{i}e\gamma^\mu$. We do not see how this comes from the Hamiltonian.

1.4 Momentum flow

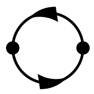
In a pair creation event the total particle number is zero, as is the total momentum. Such a process needs some energetic justification. Maybe it is a decay of a spin-0 boson into two fermions or photons. The process we have in mind looks like this:

We have momentum conservation enforced at the one vertex: $\boldsymbol{p}_1 + \boldsymbol{p}_2 = \mathbf{0}$. If we just take the absolute values using the directions defined by the diagram we have $p_1 = p_2$. Since we have no particles in total, one needs to be the anti-particle of the other one. Both particles move away from the vertex, so for anti-particles the particle number flow goes against the direction of the momentum arrow.

In the case of electron-positron-scattering one often has seen a diagram as such, where the positrons travel backward in time, so to speak:



It was asked for internal lines in the question. We can reformulate our initial process as a vacuum bubble:



The particle number must be conserved at each vertex. We can look at each vertex and interpret it like one particle is arriving, the other is leaving. The total number is conserved. Or we say that from each vertex propagates a particle and an anti-particle.

1.5 Momentum space Feynman rules for Dirac- and EM-field

The propagator for the Dirac field is by given by Peskin and Schroeder (1995, p. 118) as

$$\frac{\mathrm{i}[\not{p} + m]}{p^2 - m^2 + \mathrm{i}\epsilon}.$$

This differs from the scalar field by the factor $\not{p} + m$. The states this propagator acts are four-component spinors, the propagator therefore needs to be a 4×4 matrix. Something contracted with the Dirac matrices seems to be a good idea here. Looking at the propagator of the Dirac field in position space, we see that it exactly has the form we want here. (ibid., (3.121))

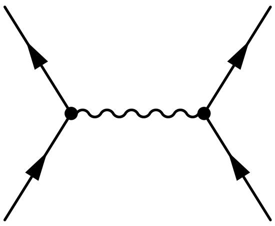
The propagator for the electromagnetic field is by given by Peskin and Schroeder (ibid., p. 123) as

$$\frac{\mathrm{i}g_{\mu\nu}}{q^2 + \mathrm{i}\epsilon}.$$

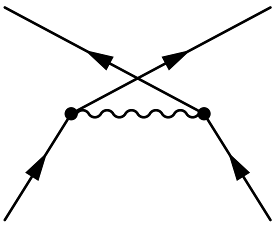
They say it is difficult to prove but easy to guess. So let us try whether we could have guessed it. The denominator makes sense since photons are massless. The imaginary unit in the numerator is also present in every other case. We have to explain the presence of the metric tensor $\boldsymbol{g}$. The four components of the electromagnetic field obey the Klein-Gordon equation individually. So the Klein-Gordon propagator is a good start. Since photons have a vector polarization and simple propagation cannot change this polarization, only states with the same polarization can propagate. The metric tensor $\boldsymbol{g}$ then creates the scalar product between the polarization vectors of the incoming and outgoing states and gives the propagation overlap. Seems legit.

1.6 Leading-order scattering

There are two possible diagrams, with the convention that time is upwards in our diagrams. The first one has a repulsion of the two fermions.



The other one has some sort of attraction.



This second diagram exchanges the two fermions. Since they are distinguishable, this would be excluded and only the first one kept.

We apply the rule and we get the following factors:

- The left incoming particle gives us a factor $u^s(\boldsymbol{k})$.
- The right one gives us $u^s(\boldsymbol{k}')$.
- The two vertices introduce a factor $-\mathrm{i}e\gamma^\mu$ each, so we get $[-\mathrm{i}e]^2\gamma^\mu\gamma^\nu$.
- The photon propagator gives us the term

  $$\frac{-\mathrm{i}g_{\mu\nu}}{\boldsymbol{q}^2 + \mathrm{i}\epsilon}$$

  where $\boldsymbol{q} := \boldsymbol{p}' - \boldsymbol{p}$ is the momentum exchanged.
- The left outgoing particle gives $\bar{u}^s(\boldsymbol{p})$.
- The right outgoing particle gives $\bar{u}^s(\boldsymbol{p}')$.

Then we have to put everything together and we arrive at the matrix element:

$$\mathrm{i}\mathcal{M} = [-\mathrm{i}e]^2 \bar{u}^s(\boldsymbol{p}')\gamma^\nu u^s(\boldsymbol{k}') \frac{-\mathrm{i}g_{\mu\nu}}{\boldsymbol{q}^2 + \mathrm{i}\epsilon} \bar{u}^s(\boldsymbol{p})\gamma^\mu u^s(\boldsymbol{k})$$

Perhaps the assigning of the momenta went wrong, but the structure of the matrix element is the same except for the spin indices that we have. They need to be summed upon, probably.

1.7 Momentum space potential

This problem covers the material also shown by Peskin and Schroeder (1995, p. 125).

We continue to use the matrix element we have just computed.

$$\mathrm{i}\mathcal{M} = [-\mathrm{i}e]^2 \bar{u}^s(\boldsymbol{p}')\gamma^\nu u^s(\boldsymbol{p}) \frac{-\mathrm{i}g_{\mu\nu}}{\boldsymbol{q}^2 + \mathrm{i}\epsilon} \bar{u}^s(\boldsymbol{k}')\gamma^\mu u^s(\boldsymbol{k})$$

We move the numbers up to the front.

$$= \frac{\mathrm{i}e^2 g_{\mu\nu}}{\boldsymbol{q}^2 + \mathrm{i}\epsilon} \bar{u}^s(\boldsymbol{p}')\gamma^\nu u^s(\boldsymbol{p}) \bar{u}^s(\boldsymbol{k}')\gamma^\mu u^s(\boldsymbol{k})$$

We look at the spinor scalar products. We look at $\nu = 0$ first.

$$\bar{u}(\boldsymbol{p}')\gamma^0 u(\boldsymbol{p}) = u^\dagger(\boldsymbol{p}')u(\boldsymbol{p})$$

The boosted spinors were given by a factor $\sqrt{\boldsymbol{p}\cdot\boldsymbol{\sigma}}$ (ibid., (3.50)). So we have

$$= \begin{pmatrix} \sqrt{\boldsymbol{p}'\cdot\boldsymbol{\sigma}}\xi' \\ \sqrt{\boldsymbol{p}'\cdot\bar{\boldsymbol{\sigma}}}\xi' \end{pmatrix}^\dagger \begin{pmatrix} \sqrt{\boldsymbol{p}\cdot\boldsymbol{\sigma}}\xi \\ \sqrt{\boldsymbol{p}\cdot\bar{\boldsymbol{\sigma}}}\xi \end{pmatrix}.$$

Since we assume $\boldsymbol{p} \to 0$, only the zeroth component in the scalar products will contribute a sizeable term. The p^0 component is the energy of the particle. However, we can directly use it as the mass in the non-relativistic limit. And although the particles are indistinguishable, they have the same mass $m \approx p^0$.

$$= \begin{pmatrix} \sqrt{p^0\boldsymbol{\sigma}^0}\xi' \\ \sqrt{p^0\boldsymbol{\sigma}^0}\xi' \end{pmatrix}^\dagger \begin{pmatrix} \sqrt{p^0\boldsymbol{\sigma}^0}\xi \\ \sqrt{p^0\boldsymbol{\sigma}^0}\xi \end{pmatrix}.$$

The $\boldsymbol{\sigma}^0$ is just the identity matrix, so this scalar product leaves us with

$$= m\xi'^\dagger\xi.$$

One can also show that the case $\nu = 1,2,3$ give a vanishing result.

Now we can use that to simplify the matrix element.

$$\begin{aligned} \mathrm{i}\mathcal{M} &= \frac{\mathrm{i}e^2 g_{\mu\nu}}{\boldsymbol{q}^2 + \mathrm{i}\epsilon} \bar{u}^s(\boldsymbol{p}')\gamma^\nu u^s(\boldsymbol{p}) \bar{u}^s(\boldsymbol{k}')\gamma^\mu u^s(\boldsymbol{k}) \\ &= \frac{\mathrm{i}e^2 g_{00}}{\boldsymbol{q}^2 + \mathrm{i}\epsilon} 4m^2 [\xi'\xi](\boldsymbol{p})[\xi'\xi](\boldsymbol{k}) \end{aligned}$$

The zeroth-zeroth element of the metric tensor is just one.

$$= \frac{\mathrm{i}e^2}{\boldsymbol{q}^2 + \mathrm{i}\epsilon} 4m^2 [\xi'\xi](\boldsymbol{p})[\xi'\xi](\boldsymbol{k})$$

The only interesting term is the fraction, so we have

$$\tilde{V}(\boldsymbol{q}) = \frac{\mathrm{i}e^2}{\boldsymbol{q}^2 + \mathrm{i}\epsilon}.$$

One can Fourier transform this back and then one would get

$$V(\boldsymbol{x}) = \frac{e^2}{4\pi}\frac{1}{r}.$$

This is a repulsive potential, since in the limit $r \to 0$, the potential energy goes up.

1.8 Antifermion

This problem covers the material also shown by Peskin and Schroeder (1995, p. 125).

For the antifermion, we need to use the spinor v instead of u. This one is made up from $(\xi, -\xi)$. This minus sign does not show up in the spinor scalar product because the $\boldsymbol{\gamma}^0$ exchanges the terms in a way that they both cancel. In the contractions that we did in the derivation, there needs to be an additional anticommutation because we want to act with a $\bar{\psi}$ to the right. This will give us a minus sign which makes the potential attractive.

References

Peskin, Michael E. and Daniel V. Schroeder (1995). *An Introduction to Quantum Field Theory*. Westview Press. ISBN: 978-0-201-50397-5.