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This is a problem set (as turned in) for the module physics755.

This problem set is not reviewed by a tutor. This is just what I have turned in.

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[disclaimer]

physics755 – Quantum Field Theory

Problem Set 8

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problem	achieved points	possible points
Wick gymnastics		8
A glimpse at the Path integral in zero dimensions		6
total		14

1 Wick gymnastics

In Equation (1) the symbol T is used as time ordering operator and the bounds of the integration. This does not make too much sense. We will use $\mathscr{T}$ for time ordering like on the last sheet.

1.1 First order

Linearization We are supposed to compute the numerator to first order in λ. The numerator is given as

$$\langle 0| \mathscr{T} \left(\phi_\text{I}(x_1)\phi_\text{I}(x_2)\phi_\text{I}(x_3)\phi_\text{I}(x_4) \exp\left(-\text{i} \int_{-T}^{T} \text{d}t \, H_\text{I}(t) \right) \right) |0\rangle$$

We expand the Hamiltonian in the interacting picture.

$$= \langle 0| \mathscr{T} \left(\phi_\text{I}(x_1)\phi_\text{I}(x_2)\phi_\text{I}(x_3)\phi_\text{I}(x_4) \exp\left(-\frac{\text{i}\lambda}{4!} \int_{-T}^{T} \text{d}t \int_{\mathbf{R}^3} \text{d}^3x \, \phi^4 \right) \right) |0\rangle$$

Since we are only interested in terms linear in λ, we linearize the exponential and use the linearity of the L^2 scalar product.

$$= \langle 0| \mathscr{T}\left(\phi_\mathrm{I}(x_1)\phi_\mathrm{I}(x_2)\phi_\mathrm{I}(x_3)\phi_\mathrm{I}(x_4)\right)|0\rangle$$
$$-\frac{\mathrm{i}\lambda}{4!}\langle 0|\mathscr{T}\left(\int_{-T}^{T}\mathrm{d}t\int_{\mathbf{R}^3}\mathrm{d}^3x\,\phi_\mathrm{I}(x_1)\phi_\mathrm{I}(x_2)\phi_\mathrm{I}(x_3)\phi_\mathrm{I}(x_4)\phi(x)\phi(x)\phi(x)\phi(x)\right)|0\rangle + \mathrm{O}(\lambda^2)$$

We use Wick's theorem in the first summand and convert the time ordering into normal ordering and contractions. Only fully contracted terms survive the vacuum (bad pun intended). We therefore only have to keep the three different contractions. The integrals and the time ordering should commute since the time ordering only applies to the integrands, not the integral signs.

$$= D(x_1-x_3)D(x_2-x_4) + D(x_1-x_4)D(x_2-x_3) + D(x_1-x_2)D(x_3-x_4)$$
$$-\frac{\mathrm{i}\lambda}{4!}\int_{-T}^{T}\mathrm{d}t\int_{\mathbf{R}^3}\mathrm{d}^3x\;\langle 0|\mathscr{T}\left(\phi_\mathrm{I}(x_1)\phi_\mathrm{I}(x_2)\phi_\mathrm{I}(x_3)\phi_\mathrm{I}(x_4)\phi(x)\phi(x)\phi(x)\phi(x)\right)|0\rangle + \mathrm{O}(\lambda^2)$$

Counting contractions We have eight field operators in the time ordering. There are $7!! = 105$ possible contractions, which are not necessarily unique. We organize those contractions into three groups, grouped by the number of self-contractions of the ϕ^4 term.

- The group with no self-contractions contains $4! = 24$ elements and they are all equal since the $\phi(x)$ terms are all equal.
- If we have one self contraction there are $4^2 \cdot 3^2/2 = 72$ possible contractions. We get to this number like so: In the beginning we have four ϕ_I and four ϕ fields to choose from. This gives 4^2. For the second contraction we can choose another ϕ_I and one ϕ field, we have three of each kind. This gives 3^2 possibilities. The two remaining ϕ_I fields are contracted with each other and so are the remaining ϕ fields. We have over counted by a factor of two since the order of performing the two contractions does not matter. We could get around this division and say that the second contraction must chose a x_j such that $j > i$ when x_i was chosen in the first contraction. We will use that below.
- If we contract all the ϕ terms among themselves, we have $3!! = 3$ possibilities to contract the ϕ_I terms. Since there are also three possibilities to contract the ϕ fields independently, we have $3^3 = 9$ contractions in this group.

Together, those are the 105 contractions that we have calculated earlier. Some of them have the same values, some are distinct. They only have the same value if they only differ in the order of the contractions of the ϕ fields.

We have computed the λ^0 order already and we will focus of the time ordering of the λ^1 order inside of the integral. We are not supposed to carry out integrations, so looking at the integrand should be sufficient. This problem demands a considerable dose of consistency, otherwise we would be lost. We call the groups with i self-contractions C_i:

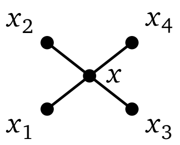
$$\langle 0|\mathscr{T}\left(\phi_\mathrm{I}(x_1)\phi_\mathrm{I}(x_2)\phi_\mathrm{I}(x_3)\phi_\mathrm{I}(x_4)\phi(x)\phi(x)\phi(x)\phi(x)\right)|0\rangle = C_0 + C_1 + C_2.$$

Then we will look into each group separately.

No self contractions So we start with the terms in the group without self contractions. Those terms are all equal. Again, we only consider fully contracted terms here, since the vacuum expectation values of all other normal ordered products is zero. We choose one representative contraction and multiply it by 4!.

$$C_0 = 4! \langle 0| \phi_\mathrm{I}(x_1)\phi_\mathrm{I}(x_2)\phi_\mathrm{I}(x_3)\phi_\mathrm{I}(x_4)\phi(x)\phi(x)\phi(x)\phi(x) |0\rangle = 4! \prod_{i=1}^{4} D(x_i - x)$$

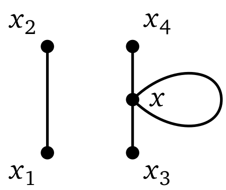
This can be pictured as the following diagram:



One self contraction We have to come up 72 contractions here. However, there are only six unique configuration. This is easiest to see since there are only six ways to form one contraction of the ϕ_I uniquely. The duplicity will be $72/6 = 12$ for each term. There will be a $D(0)$ from the single contraction of the ϕ^4 term.

$$\begin{aligned} C_2 = 12\big[&D(x_1 - x_2)D(x_3 - x)D(x_4 - x) + D(x_1 - x_3)D(x_2 - x)D(x_4 - x) \\ &+ D(x_1 - x_4)D(x_2 - x)D(x_3 - x) + D(x_2 - x_3)D(x_1 - x)D(x_4 - x) \\ &+ D(x_2 - x_4)D(x_1 - x)D(x_3 - x) + D(x_3 - x_4)D(x_1 - x)D(x_2 - x)\big] D(0) \end{aligned}$$

In total, there are $12 \cdot 6 = 72$ terms, this is fine. Sorry for not typesetting the contractions here, they just eat too much time. Creating Feynman diagrams did cost some time as well. I guess I am hypocritical here :-). Anyway, the first summand can be pictured like this:

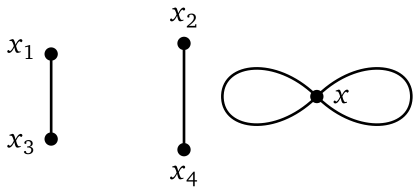


All the other graphs are variations of this.

Two self contractions As written earlier, the contractions where the ϕ^4 term is fully contracted with itself are independent of the contractions of the ϕ_I with each other. We see the zeroth order term with an additional factor from the ϕ^4 term. There are three different contractions within ϕ^4, so we get an additional factor three.

$$C_2 = 3\big[D(x_1 - x_3)D(x_2 - x_4) + D(x_1 - x_4)D(x_2 - x_3) + D(x_1 - x_2)D(x_3 - x_4)\big] D(0)^2$$

The first summand can be pictured like this:



This is the first time that we have a disconnected piece in our diagram.

End result Now we just have to combine all three terms together to get the desired result.

$$
\begin{aligned}
&-\frac{\mathrm{i}\lambda}{4!}\int_{-T}^{T}\mathrm{d}t\int_{\mathbf{R}^3}\mathrm{d}^3x\ \langle 0|\mathscr{T}\left(\phi_\mathrm{I}(x_1)\phi_\mathrm{I}(x_2)\phi_\mathrm{I}(x_3)\phi_\mathrm{I}(x_4)\phi(x)\phi(x)\phi(x)\phi(x)\right)|0\rangle \\
&=-\frac{\mathrm{i}\lambda}{4!}\int_{-T}^{T}\mathrm{d}t\int_{\mathbf{R}^3}\mathrm{d}^3x\ \langle 0|\Bigg[4!\prod_{i=1}^{4}D(x_i-x) \\
&\quad+12\big[D(x_1-x_2)D(x_3-x)D(x_4-x)+D(x_1-x_3)D(x_2-x)D(x_4-x) \\
&\quad+D(x_1-x_4)D(x_2-x)D(x_3-x)+D(x_2-x_3)D(x_1-x)D(x_4-x) \\
&\quad+D(x_2-x_4)D(x_1-x)D(x_3-x)+D(x_3-x_4)D(x_1-x)D(x_2-x)\big]D(0) \\
&\quad+3\big[D(x_1-x_3)D(x_2-x_4)+D(x_1-x_4)D(x_2-x_3)+D(x_1-x_2)D(x_3-x_4)\big]D(0)^2\Bigg]|0\rangle
\end{aligned}
$$

Each term has four propagators in it, which should be the case.

1.2 Denominator and disconnected pieces

We have a disconnected piece in only one of the diagrams, the one where the ϕ^4 term is completely contracted with itself.

Basically we need to reproduce the material from Peskin and Schroeder (1995, pp. 96-98) here and look at all orders and see that the disconnected parts can be factored out. Looking at the first order only does not suffice.

1.3 Odd number of fields

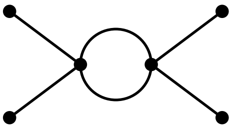
With an odd number of fields we cannot fully contract the fields. The one field operator that is left will be normal ordered and annihilate the vacuum. Those Green's functions are zero.

1.4 Second order diagrams

In the second order we have two internal vertices, v_1 and v_1 that we need to integrate over. There are still four external points, the x_i. The wording in the problem statement says that we shall draw all the diagrams *contributing* to order λ^2, which means that we are not interested in disconnected pieces.

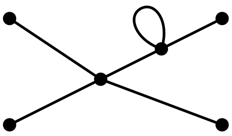
If we factor out the disconnected pieces, we have the same contribution like above, just with an additional $D(0)^2$ from the second internal vertex. We will not draw those, since the connected part only has one internal vertex and is nothing new.

1. The first diagram has one self-contraction of the internal vertices. We cannot have to self-contractions since that would give a disconnected "head with ears" diagram.

   Now we have to collect factors for the symmetries: There are no lines that start and end on the same vertex. The two lines connecting the internal vertices can be interchanged and give a factor two. This diagram has $S = 2$.

2. The second option that we have is to connect three external vertices to an internal one and self-contract the other internal vertex with itself.

   This diagram has a line that ends on the vertex it started from. Other than that, there is nothing to exchange, so we have $S = 2$.

3. We can connect one external vertex with another external one and get one more self-contraction of the internal vertices.

   This has two closed loops and we have a symmetric factor $S = 4$ here.

4. We can have the self-contraction on the other side as well.

   Here we have a factor 2^2 for the two closed loops. Is this an instance of equivalent subgraphs and we get another 2! from that? If so, the symmetry factor is $S = 8$.

2 A glimpse at the Path integral in zero dimensions

2.1 Linear coupling

Integral The integral can be solved by completing the square in the exponential function.

$$Z_0 = \int_{-\infty}^{\infty} \mathrm{d}x \, \exp\left(-\frac{m^2}{2}x^2 + Jx\right)$$

We move the Jx term into the bracket.

$$= \int_{-\infty}^{\infty} \mathrm{d}x \, \exp\left(-\frac{m^2}{2}\left[x^2 - \frac{2}{m^2}Jx\right]\right)$$

Then we complete the square.

$$\begin{aligned}
&= \int_{-\infty}^{\infty} \mathrm{d}x \, \exp\left(-\frac{m^2}{2}\left[x^2 - \frac{2}{m^2}Jx + \frac{J^2}{m^4} - \frac{J^2}{m^4}\right]\right) \\
&= \int_{-\infty}^{\infty} \mathrm{d}x \, \exp\left(-\frac{m^2}{2}\left[x^2 - \frac{2}{m^2}Jx + \frac{J^2}{m^4}\right] + \frac{J^2}{2m^2}\right) \\
&= \int_{-\infty}^{\infty} \mathrm{d}x \, \exp\left(-\frac{m^2}{2}\left[x - \frac{J}{m^2}\right]^2 + \frac{J^2}{2m^2}\right)
\end{aligned}$$

The term constant in x can be split off and moved in front of the integral.

$$= \exp\left(\frac{J^2}{2m^2}\right) \int_{-\infty}^{\infty} \mathrm{d}x \, \exp\left(-\frac{m^2}{2}\left[x - \frac{J}{m^2}\right]^2\right)$$

The integration bounds do not change with a finite shift. We shift x such that the recover the Gaussian integral formula.

$$= \exp\left(\frac{J^2}{2m^2}\right) \int_{-\infty}^{\infty} \mathrm{d}y \, \exp\left(-\frac{m^2}{2}y^2\right)$$

Now we can apply the formula and yield the result.

$$= \exp\left(\frac{J^2}{2m^2}\right) \frac{\sqrt{2\pi}}{m}$$

Derivative The quantization of the points here should give half a point for this problem. Since having the result should be more than half, we should get the point, although the derivation is missing. We computed the first ten derivatives with Mathematica and read off the pattern. This is the result:

$$\frac{\partial^n Z}{\partial J^n}(0) = [n-1]!! \frac{\sqrt{2\pi}}{m^{n+1}} \times \begin{cases} 1 & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

2.2 Nonlinear perturbation

The integration in Equations (11) and (12) on the problem set is over the measure of $\mathbf{R}^4$, which probably is a mistake, since we are in the zero dimensional case and the space of functions on the one point set only has one dimension, it can be parametrized by $\mathbf{R}$. In the path integral formalism, each degree of freedom corresponds to one integration. Having four integrations with $\mathrm{d}x^4$ would mean that there are exactly four degrees of freedom. For a field theory, this does not sound sensible.

We will assume that the integration $\int \mathrm{d}x$ was meant and that the power of four just crept in since the perturbation also has a power of four now.

First we derive the expansion of a fraction:

$$\frac{a + \lambda b}{c + \lambda d} = \frac{a}{c} + \frac{bc - ad}{c^2}\lambda + \mathrm{O}(\lambda^2).$$

The linearization of the exponential in λ will give us

$$1 + \frac{\lambda}{4!}x^4.$$

We can therefore write:

$$a = \int_{-\infty}^{\infty} \mathrm{d}x\, x^4 \exp\left(-\frac{1}{2}x^2\right) = 3\sqrt{2\pi}$$
$$b = \frac{1}{4!}\int_{-\infty}^{\infty} \mathrm{d}x\, x^8 \exp\left(-\frac{1}{2}x^2\right) = \frac{105}{4!}\sqrt{2\pi}$$
$$c = \int_{-\infty}^{\infty} \mathrm{d}x\, \exp\left(-\frac{1}{2}x^2\right) = \sqrt{2\pi}$$
$$d = \frac{1}{4!}\int_{-\infty}^{\infty} \mathrm{d}x\, x^4 \exp\left(-\frac{1}{2}x^2\right) = \frac{3}{4!}\sqrt{2\pi}$$

Using those components, we can build up our result to first order in λ.

$$\langle \Omega | x^4 | \Omega \rangle = \frac{a}{c}\frac{bc - ad}{c^2}\lambda + \mathrm{O}(\lambda^2)$$

We insert a, b, c, and d. The $\sqrt{2\pi}$ cancels in all terms and just the factors in front are left. Only one of the factors contains the term with the factorial, so we can pull that up front.

$$= 3 + \frac{105 \cdot 1 - 3 \cdot 3}{4!}\lambda + \mathrm{O}(\lambda^2)$$
$$= 3 + 4\lambda + \mathrm{O}(\lambda^2)$$

2.3 Diagrams

Missing

References

Peskin, Michael E. and Daniel V. Schroeder (1995). *An Introduction to Quantum Field Theory*. Westview Press. ISBN: 978-0-201-50397-5.