Disclaimer

This is a problem set (as turned in) for the module physics755.

This problem set is not reviewed by a tutor. This is just what I have turned in.

All problem sets for this module can be found at
http://martin-ueding.de/de/university/msc_physics/physics755/.

[disclaimer]

physics755 – Quantum Field Theory

Problem Set 7

Martin Ueding
mu@martin-ueding.de

Oleg Hamm

2015-06-08

Group Tuesday – Ripunjay Acharya

problem	achieved points	possible points
PCT and all that		10
Wick's theorem		5
total		15

1 PCT and all that

1.1 Table

Here we have to compute $4 \cdot 6 = 24$ transformation properties. We get whopping four points for this problem. So this means that we have to write about four pages to get everything done. The ratio seems rather bad here as we only finish on page 7 with the very first problem.

Often we have to use that the symmetric part (in idempotent form) of $\boldsymbol{\gamma} \otimes \boldsymbol{\gamma}$ is related to the metric tensor:

$$\boldsymbol{\gamma}^{(\mu}\boldsymbol{\gamma}^{\nu)} = \eta^{\mu\nu}\mathbf{1}_4.$$

In particular this means that

$$[\boldsymbol{\gamma}^{\mu}]^2 = \eta^{\mu\mu}\mathbf{1}_4.$$

Why is the quantity $\frac{\mathrm{i}}{2}[\boldsymbol{\gamma}^{\mu}, \boldsymbol{\gamma}^{\nu}]$ now called $\boldsymbol{\sigma}^{\mu\nu}$. We had defined this earlier to be the commutator of the Pauli matrices. The commutator of the Dirac matrices was called $S^{\mu\nu}$. Either way, we will use the notation from this problem set.

Parity Since there are six tensorial quantities, we will just number them for each transformation operator.

1. To transform with the parity, we just sandwich the expression into Ps and use that P^2 is the identity since it comes from the cyclic group $\mathbf{Z}_2$. We added two independent $\mathbf{Z}_2$ to the universal covering group we get from the exponential map of the Lie algebra $\mathfrak{so}(1,3)$ to reach the whole Lorentz group.

   $$\bar\psi(t,\boldsymbol{x})\psi(t,\boldsymbol{x}) \mapsto P\bar\psi(t,\boldsymbol{x})\psi(t,\boldsymbol{x})P$$

   We introduce a $P^2 = 1$ in the middle.

   $$= P\bar\psi(t,\boldsymbol{x})PP\psi(t,\boldsymbol{x})P$$

   Now we can use the transformation rule given in Equation (2) on the problem set. The phase η here is not the metric tensor η which shows up in the table given on the problem set. It neither is the rapidity η which was used on a different sheet. This is ambiguous, but luckily, there is always context around, to make it clear. Right?

   $$= \eta^*\bar\psi(t,-\boldsymbol{x})\boldsymbol{\gamma}^0\eta\boldsymbol{\gamma}^0\psi(t,-\boldsymbol{x})$$

   The phase η is a complex number and therefore commutes with the matrices.

   $$= \eta^*\eta\bar\psi(t,-\boldsymbol{x})\boldsymbol{\gamma}^0\boldsymbol{\gamma}^0\psi(t,-\boldsymbol{x})$$

   The modulus of the phase is unity, so we can drop this. The Dirac matrices are their own inverses, so we can drop those as well. We are left with the Lorentz scalar as the transformed point.

   $$= \bar\psi(t,-\boldsymbol{x})\psi(t,-\boldsymbol{x})$$

   To make it more apparent that we transformed the whole thing to one new coordinate, we can write it like this:

   $$= [\bar\psi\psi](t,-\boldsymbol{x}).$$

   Then the additional factor that we get is just $+1$. If we keep up this pace we will end up with a quarter of a point per page, so we need to pick up the pace a bit.

2. The next one, $\mathrm{i}\bar\psi\boldsymbol{\gamma}^5\psi$, has an imaginary unit in it, as well as a $\boldsymbol{\gamma}^5$. The parity operator will give us $\boldsymbol{\gamma}^0$s, which anticommute with the $\boldsymbol{\gamma}^5$, so that will give us the minus sign.

3. The bilinear $\bar\psi\boldsymbol{\gamma}^\mu\boldsymbol{\gamma}^5\psi$ will transform differently depending on the μ. If $\mu = 0$, it commutes with the $\boldsymbol{\gamma}^0$ that we get from the parity transformation. If $\mu \neq 0$, we have to perform an anticommutation. Hence a factor of $\eta^{\mu\nu}$.

4. Adding another $\boldsymbol{\gamma}^5$ to the bilinear means another anticommutation compared to the previous one and therefore a factor of $-\eta^{\mu\mu}$.

5. We expand the transformed bilinear:

   $$\bar\psi\boldsymbol{\gamma}^0\boldsymbol{\sigma}^{\mu\nu}\boldsymbol{\gamma}^0\psi = \frac{\mathrm{i}}{2}\bar\psi\boldsymbol{\gamma}^0[\boldsymbol{\gamma}^\mu,\boldsymbol{\gamma}^\nu]\boldsymbol{\gamma}^0\psi = \frac{\mathrm{i}}{2}\bar\psi\left[\boldsymbol{\gamma}^0\boldsymbol{\gamma}^\mu\boldsymbol{\gamma}^\nu\boldsymbol{\gamma}^0 - \boldsymbol{\gamma}^0\boldsymbol{\gamma}^\nu\boldsymbol{\gamma}^\mu\boldsymbol{\gamma}^0\right]\psi.$$

One can see that it takes two anticommutations to get the $\boldsymbol{\gamma}^0$ to cancel the other one. Those anticommutations introduce factors of $\eta^{\mu\mu}$ and $\eta^{\nu\nu}$ respectively. Therefore the factor that we end up with is the product of the two.

6. We apply the chain rule here and since we map $\boldsymbol{x} \mapsto -\boldsymbol{x}$ the chain rule will just give us a minus sign in the spatial components. This is expressed in $\eta^{\mu\mu}$.

Time reversal We have to do the same thing for the time reversal operation.

1. We do the first one more explicit as before. We start with the sandwich between the time reversal operators.

$$\bar{\psi}\psi \mapsto T\bar{\psi}\psi T$$

We add a unit operator in between the field operators.

$$= T\bar{\psi}T^\dagger T\psi T$$

To use the formula that is given in Equation (5) we must assume that $T^\dagger = T$. Or are we really using that T^2 must be the identity and not introduce a $T^\dagger$ at all? Either way, we must write

$$= T\bar{\psi}TT\psi T.$$

Now we apply Equation (5) from the problem set.

$$= -\bar{\psi}(-t, \boldsymbol{x})\boldsymbol{\gamma}^1\boldsymbol{\gamma}^3\boldsymbol{\gamma}^1\boldsymbol{\gamma}^3\psi(-t, \boldsymbol{x})$$

It takes one anticommutation in the middle.

$$= \bar{\psi}(-t, \boldsymbol{x})\boldsymbol{\gamma}^3\boldsymbol{\gamma}^1\boldsymbol{\gamma}^1\boldsymbol{\gamma}^3\psi(-t, \boldsymbol{x})$$

We can collapse the Dirac matrices and have a compact result. Both pairs of Dirac matrices give a minus, but that does not change the net sign.

$$= [\bar{\psi}\psi](-t, \boldsymbol{x})$$

2. The pseudo scalar has another $\boldsymbol{\gamma}^5$ which means another anticommutation is needed. This makes the factor -1.

3. The vector is also a bit tricky, so we need several steps here as well.

$$\bar{\psi}\boldsymbol{\gamma}^\mu\psi \mapsto T\bar{\psi}\boldsymbol{\gamma}^\mu\psi T$$

We add unit operators in between the elements.

$$= T\bar{\psi}TT^\dagger\boldsymbol{\gamma}^\mu T^\dagger T\psi T$$

We use the given transformation formulas on the field operators. At the same time, we use that γ^μ is a hermitian matrix and factor out the hermitian conjugate.

$$= -\bar{\psi}(-t, \boldsymbol{x})\gamma^1\gamma^3 \left[T\gamma^\mu T\right]^\dagger \gamma^1\gamma^3\psi(-t, \boldsymbol{x})$$

Next we use the hint which tells us that time reversal complex conjugates the Dirac matrices: γ^μ will be complex conjugated. For an hermitian matrix, this is the same as the transpose. For the cases $\mu = 1$ and $\mu = 3$ the transpose will have an additional minus sign compared to the matrix itself. However, these are also the cases where we need one anticommutation less to get the γ^μ across the $\gamma^1\gamma^3$. This effect therefore cancels. The $\mu = 0$ case has no sign change associated with it whereas the other cases have a sign change.

$$= -\eta^{\mu\mu}\bar{\psi}(-t, \boldsymbol{x})\gamma^1\gamma^3\gamma^1\gamma^3\boldsymbol{\sigma}^{\mu\nu}\psi(-t, \boldsymbol{x})$$

We need to anticommute the Dirac matrices to form nested pairs. This will change the sign again.

$$= \eta^{\mu\mu}\bar{\psi}(-t, \boldsymbol{x})\gamma^1\gamma^3\gamma^3\gamma^1\boldsymbol{\sigma}^{\mu\nu}\psi(-t, \boldsymbol{x})$$

The two nested pairs will both give a sign change this cancels in the end result.

$$= \eta^{\mu\mu}\bar{\psi}(-t, \boldsymbol{x})\boldsymbol{\sigma}^{\mu\nu}\psi(-t, \boldsymbol{x})$$

4. The pseudovector works in the same way. We have an additional γ^5 here. Since the time reversal generates two Dirac matrices additionally, we need two anticommutations to get them across the γ^5 this time. No sign change happens because of this.

5. The tensor requires care of all the different cases. We start by expanding the Pauli matrix tensor.

$$T\bar{\psi}\boldsymbol{\sigma}^{\mu\nu}\psi T = T\bar{\psi}\frac{\mathrm{i}}{2}\left[\gamma^\mu, \gamma^\nu\right]\psi T$$

We add more time reflection operators.

$$= T\bar{\psi}TT\frac{\mathrm{i}}{2}\left[\gamma^\mu, \gamma^\nu\right]TT\psi T$$

There are a lot of sign changes here, so we want to be careful. The first one comes from moving T past the imaginary unit.

$$= -T\bar{\psi}T\frac{\mathrm{i}}{2}T\left[\gamma^\mu, \gamma^\nu\right]TT\psi T$$

Now we can apply the time reversal on both the spinors and the commutator of Dirac matrices.

$$= \bar{\psi}\gamma^1\gamma^3\frac{\mathrm{i}}{2}\left[\gamma^{\mu *}, \gamma^{\nu *}\right]\gamma^1\gamma^3\psi$$

We perform one anticommutation to the matrices in the right order to cancel each other once we have moved them through the commutator. This gives us another minus sign.

$$= -\bar{\psi}\gamma^1\gamma^3\frac{\mathrm{i}}{2}[\gamma^{\mu *}, \gamma^{\nu *}]\gamma^3\gamma^1\psi$$

Only the Dirac matrix γ^2 is imaginary, all the other ones are real in the Weyl representation. The complex conjugation therefore gives us a minus sign whenever $\mu = 2$ or $\nu = 2$. It is symmetric in μ and ν, so we only need to look at six distinct cases. We start with $\mu = 0$ and $\nu = 1$. We need one anticommutation for γ^1 and two anticommutations for γ^3. No minus sign will come from the complex conjugation. We therefore write this as

$$N^{\mu\nu} = N_{\gamma^1} + N_{\gamma^3} + N_* = 1 + 2 + 0 = 3$$

sign changes in total.

Going through all the cases in this fashion, we end up with the following symmetric matrix:

$$\boldsymbol{N} = \begin{pmatrix} 1+2+0=3 & 2+2+1=5 & 2+1+0=3 \\ & 1+2+1=4 & 1+1+0=2 \\ & & 2+1+1=4 \end{pmatrix}.$$

This can be summed up in

$$= -\eta^{\mu\mu}\eta^{\nu\nu}\bar{\psi}\gamma^1\gamma^3\gamma^3\gamma^1\gamma^{\mu\nu}\psi.$$

Now we can remove the pairs of Dirac matrices. Each pair will give a minus sign, so the total sign does not change.

$$= -\eta^{\mu\mu}\eta^{\nu\nu}\bar{\psi}\psi$$

6. The partial derivative transform with the chain rule again. We only change the sign in the time direction, so the spatial parts are left intact. With our convention of the signature of the metric, this will be $-\eta^{\mu\mu}$.

Charge conjugation The charge conjugation includes spatial and temporal Dirac matrices in the transformed results, so there will not be any distinction on the indices μ or ν in the final results since there are the same number of anticommutations each.

1. We start with the scalar.

$$\bar{\psi}\psi \mapsto C\bar{\psi}\psi C$$

One needs another unit operator in the middle to use the transformation rule.

$$= C\bar{\psi}CC\psi C$$

We use the transformation given in Equation (3).

$$= -\left[\gamma^0\gamma^2\psi\right]^{\mathrm{T}}\left[\bar{\psi}\gamma^0\gamma^2\right]^{\mathrm{T}}$$

We take the transpose of the whole thing to get the order back in the normal one. In this step we change the order of the field operators which will give us another minus sign from the fermionic anticommutation.

$$= \left[\bar{\psi}\gamma^0\gamma^2\gamma^0\gamma^2\psi\right]^{\mathrm{T}}$$

The expression in the bracket is a scalar, we can therefore drop the transpose.

$$= \bar{\psi}\gamma^0\gamma^2\gamma^0\gamma^2\psi$$

It takes one anticommutation to order the Dirac matrices into nested pairs.

$$= -\bar{\psi}\gamma^0\gamma^2\gamma^2\gamma^0\psi$$

The pair γ^2 will give us a minus sign, the pair with γ^0 will not. This is another sign change.

$$= \bar{\psi}\psi$$

2. The additional γ^5 in the pseudo scalar will lead to two additional anticommutations which do not change the sign in the end.
3. The additional γ^μ will cost one anticommutation with the Dirac matrices already in the expression. As seen before the transpose of the Dirac matrix gives additional minus signs for $\mu = 1$ and $\mu = 3$. Here the Dirac matrices for $\mu = 0$ and $\mu = 2$ are present, so the minus sign does not cancel but happen in every of the four possible terms.
4. An additional γ^5 in the pseudo vector again flips the sign with respect to the vector.
5. The tensor bilinear under charge conjugation is next.

$$\bar{\psi}\sigma^{\mu\nu}\psi \mapsto C\bar{\psi}\sigma^{\mu\nu}\psi C$$

We expand the matrix commutator and add more conjugation operators. We use the property that the charge conjugation operator commutes with all the Dirac matrices.

$$= C\bar{\psi}C\frac{\mathrm{i}}{2}[\gamma^\mu, \gamma^\nu]C\psi C$$

Then we write out the transformed spinors.

$$= -\left[\gamma^0\gamma^2\psi\right]^{\mathrm{T}}\frac{\mathrm{i}}{2}[\gamma^\mu, \gamma^\nu]\left[\bar{\psi}\gamma^0\gamma^2\right]^{\mathrm{T}}$$

We do the same thing as with the scalar, to get everything into a transpose bracket. After that, we can drop it since the transpose of a scalar still is a scalar.

$$= -\bar{\psi}\gamma^0\gamma^2\frac{\mathrm{i}}{2}[\gamma^\mu,\gamma^\nu]^\mathrm{T}\gamma^0\gamma^2\psi$$

We perform the anticommutation of the Dirac matrices now and remove the leading minus sign.

$$= \bar{\psi}\gamma^0\gamma^2\frac{\mathrm{i}}{2}[\gamma^\mu,\gamma^\nu]^\mathrm{T}\gamma^2\gamma^0\psi$$

The transpose of the commutator will give introduce additional minus signs for $\mu = 2$ or $\nu = 2$. In those cases we need one anticommutation less, so there is no net sign change when this is the case. If the other index is 0, one should need one less anticommutation and therefore incur a sign change.

Once everything is brought through the commutator, the nested pairs of Dirac matrices will give an additional minus sign.

6. Derivatives of spacetime have nothing to do with charge, so no chain rule applies and the sign stays as it is.

All transformations This one is easy: Just multiply all the factors from parity, time reversal and charge conjugation and you get the factor from the combined CPT transformation.

1.2 Short answers

There are six questions. We will number them to aid navigation.

1. Angular momentum probably is the Hodge dual of a 2-form, which does changes the sign twice under parity. Just like the magnetic field $\boldsymbol{B}$, which is the Hodge dual of a 2-form. It only feels strange to have vectorial quantities that are invariant under parity because they are constructed from forms and we then forget to mention that. This class of vectors is called pseudo vectors or axial vectors and usually arises from a cross product, which is related to the wedge product, but only in $\mathbf{R}^3$.

2. The direction of propagation flips, the angular momentum does not. Together the helicity flips and therefore is called a pseudo scalar.

3. Angular momentum changes its sign under time inversion. So the helicity does not change. That would be really strange actually, if the helicity would change in time reversal. Think of a falling gyro which has a meridian marked. The helix described by this thing should not change with time reversal.

4. Which representation is meant? We assume that it is the natural one on the four-vectors of Minkowski space.

   The generators for boosts would have a sign change under both parity and time inversion. Write such a generator $\boldsymbol{K}$ with either transformation: $P\boldsymbol{K}P$. The one parity operator will flip the sign in the spatial columns of $\boldsymbol{K}$, the other one will flip the sign in the spatial rows. Together, only the signs of the K^{0i} components are flipped. The time reversal will slip the temporal row and column, yielding the same result.

The generators for rotations would not change under either transformation. The parity will change the spatial components twice. The rotation generator $\boldsymbol{J}$ only has spatial components that are nonzero, so it does not change at all. The time reversal will not change the spatial components at all, so $\boldsymbol{J}$ is invariant here as well.

5. The representations of the Group SO(1,3) that we know are (a) SO(1,3) matrices that act on the four-vectors of Minkowski space, (b) differential operators on the scalar functions, (c) 2×2 matrices on Weyl spinors and (d) 4×4 matrices on the Dirac spinors. Representation (d) is just representation (c) repeated twice in a block diagonal form.

6.

1.3 Complex Klein-Gordon field

The real field $\phi(\boldsymbol{x})$ is given as:

$$\phi(\boldsymbol{x}) = \int \frac{\mathrm{d}^3 p}{[2\pi]^3} \frac{1}{\sqrt{E_{\boldsymbol{p}}}} \left[a_{\boldsymbol{p}} \exp(-\mathrm{i}\boldsymbol{p} \cdot \boldsymbol{x}) + a_{\boldsymbol{p}}^\dagger \exp(\mathrm{i}\boldsymbol{p} \cdot \boldsymbol{x}) \right].$$

Since we have a complex Klein-Gordon field we need two different sets of ladder operators. We call the second one b and $b^\dagger$.

$$\phi(\boldsymbol{x}) = \int \frac{\mathrm{d}^3 p}{[2\pi]^3} \frac{1}{\sqrt{E_{\boldsymbol{p}}}} \left[a_{\boldsymbol{p}} \exp(-\mathrm{i}\boldsymbol{p} \cdot \boldsymbol{x}) + b_{\boldsymbol{p}}^\dagger \exp(\mathrm{i}\boldsymbol{p} \cdot \boldsymbol{x}) \right].$$

We can add a time dependence by making this Schrödinger picture operator into a Heisenberg picture operator.

$$\phi_{\mathrm{H}}(t, \boldsymbol{x}) = \int \frac{\mathrm{d}^3 p}{[2\pi]^3} \frac{1}{\sqrt{E_{\boldsymbol{p}}}} U^\dagger(t) \left[a_{\boldsymbol{p}} \exp(-\mathrm{i}\boldsymbol{p} \cdot \boldsymbol{x}) + b_{\boldsymbol{p}}^\dagger \exp(\mathrm{i}\boldsymbol{p} \cdot \boldsymbol{x}) \right] U(t)$$

This does not help us much, yet.

We need to write this more compact, with an explicit time dependence in the exponentials. Peskin and Schroeder (1995, p. 25) look at the commutator of the Hamiltonian operator H with the annihilation operator $a_{\boldsymbol{p}}$ and use this. Here we focus on a only. We will show the commutator first.

$$\left[H, a_{\boldsymbol{p}}\right] = \int \frac{\mathrm{d}^3 p'}{[2\pi]^3} \omega_{\boldsymbol{p}'} \left[a_{\boldsymbol{p}'}^\dagger a_{\boldsymbol{p}'}, a_{\boldsymbol{p}} \right]$$

The annihilation operator commutes with itself, we can extract that from the commutator.

$$= \int \frac{\mathrm{d}^3 p'}{[2\pi]^3} \omega_{\boldsymbol{p}'} \left[a_{\boldsymbol{p}'}^\dagger, a_{\boldsymbol{p}} \right] a_{\boldsymbol{p}'}$$

By definition, this is a Dirac δ-distribution. Executing the integral just gives us

$$= \omega_{\boldsymbol{p}} a_{\boldsymbol{p}}.$$

This result can then be used to remove the unitary time evolution operators U and replace them with a straight time dependence in the exponential by using this relation n times in each summand of the exponential. The same procedure can be done for $b^\dagger$ as well, the sign will be the opposite, though.

We now have

$$\phi_{\mathrm{H}}(t, \boldsymbol{x}) = \int \frac{\mathrm{d}^3 p}{[2\pi]^3} \frac{1}{\sqrt{E_{\boldsymbol{p}}}} \left[a_{\boldsymbol{p}} \exp\left(\mathrm{i}[E_{\boldsymbol{p}} t - \boldsymbol{p} \cdot \boldsymbol{x}]\right) + b_{\boldsymbol{p}}^\dagger \exp\left(-\mathrm{i}[E_{\boldsymbol{p}} t - \boldsymbol{p} \cdot \boldsymbol{x}]\right) \right].$$

We finally have the field expressed in terms of creation and annihilation operators with explicit time dependence. One could use four-vectors to compress the notation a bit, but we need to look at space and time individually in the next parts anyway.

Parity The action of our transformation operators shall be given in terms of their action on the ladder operators. To flip $\boldsymbol{x}$ here, we should map $a_{\boldsymbol{p}} \mapsto a_{-\boldsymbol{p}} = a_{\boldsymbol{p}}^\dagger$ and then use the symmetry of the integral with respect to p. The exponential would have the exact same value if one looks at $\boldsymbol{x} \mapsto -\boldsymbol{x}$, which is the desired transformation.

Time reversal The time reversal is trickier, as expected. We need to exchange the exponentials and also perform the parity. So we have $a_{\boldsymbol{p}} \mapsto b_{-\boldsymbol{p}}^\dagger = b_{\boldsymbol{p}}$ and the corresponding other way around.

Charge conjugation The complex Klein-Gordon field has distinct particles and antiparticles. To conjugate the charge, we need to exchange those two types of particles. This can be done by exchanging $a \mapsto b$ and $b \mapsto a$.

Transformation properties of current

- The transformation with the parity is:

  $$J^\mu \mapsto P J^\mu P$$

  We add more parity operators in between. To make it easier on the eye, we added some spacing between the individual parts.

  $$= \mathrm{i}\,[P\phi^* P \; P\partial^\mu P \; P\phi P - \text{c.c.}]$$

  Now we can apply the given transformation properties given on the problem set to each term. The partial derivative transforms like we have computed in problem 1.1.

  $$= \mathrm{i}\,[\phi^*(t, -\boldsymbol{x}) \; \eta^{\mu\mu}\partial^\mu \; \phi(t, -\boldsymbol{x}) - \text{c.c.}]$$

We factor out J^μ again to make the additional factors visible.

$$= \eta^{\mu\mu} J^\mu(t, -\boldsymbol{x})$$

- The charge conjugation should flip the sign of the current. And indeed it does. The charge density is purely real, taking the complex conjugate does not reverse the sign. We will show that in detail now.

  $$J^\mu \mapsto C J^\mu C$$

  We insert J^μ explicitly and add more charge conjugation operators.

  $$= C\mathrm{i}C \left[C\phi^* C \; C\partial^\mu C \; C\phi C - \text{c.c.} \right]$$
  $$= -\mathrm{i} \left[\phi \partial^\mu \phi^* - \text{c.c.} \right]$$

  We exchange the two summands in the brackets to get back to J^μ and remove the minus sign.

  $$= \mathrm{i} \left[\phi^* \partial^\mu \phi - \text{c.c.} \right]$$

  And this is just the current density that we have started with.

  $$= J^\mu(t, \boldsymbol{x})$$

- The time reversal works similarly, except that there is a sign change from the partial derivative compared to the parity, see the table from problem 1.1. We also get a sign change from the imaginary unit and the fields get complex conjugated. This just changes the order and also gives a sign change with respect to the original current density.

  $$J^\mu \mapsto T J^\mu T$$

  We add more time reversal operators in between, also around the imaginary unit.

  $$= T\mathrm{i}T \; \left[T\phi^* T \; T\partial^\mu T \; T\phi T - \text{c.c.} \right]$$

  The imaginary unit gives us an additional minus sign. The fields are taken at $-t$. The partial derivative gives a factor of $-\eta^{\mu\mu}$.

  $$= \eta^{\mu\mu} J^\mu(-t, \boldsymbol{x})$$

1.4 CPT invariance

When we computed the table with the transformation properties we have seen that both scalars and pseudo-scalars composed of ψ are invariant under CPT. Something only is a Lorentz scalar if it transforms as such. And scalars do not change their sign under parity or time reversal. The charge conjugation could give us the complex conjugate in principle. However, hermitian scalars must be real, therefore the complex conjugation does not change anything. All in all, hermitian (and therefore real) scalars are invariant under CPT.

2 Wick's theorem

This problems introduces another overload of the letter T, it now is both time ordering and time reversal operator. We have yet to define another T to make it even with the thrice defined η. To make it easier for the reader, we chose to introduce script letters here, mostly because LaTeX can do this and they are not used yet. The time ordering will be denoted with $\mathscr{T}$ and the normal ordering with $\mathscr{N}$.

2.1 Two fields

We have to look at vacuum matrix elements here. Peskin and Schroeder (1995, Section 4.3) do this as well, and we need it to get a Feynman propagator in the end. We start by expanding the time ordering in terms of Heaviside step functions.

$$\langle 0 | \mathscr{T}(\phi_1 \phi_2) | 0 \rangle = \Theta(x_1^0 - x_2^0) \langle 0 | \phi_1 \phi_2 | 0 \rangle + \Theta(x_2^0 - x_1^0) \langle 0 | \phi_2 \phi_1 | 0 \rangle$$

We can identify this with the definition of the Feynman propagator.

$$= D_\mathrm{F}(\boldsymbol{x} - \boldsymbol{y})$$

The vacuum state is normalized, we can add a bracket around this scalar value.

$$= \langle 0 | D_\mathrm{F}(\boldsymbol{x} - \boldsymbol{y}) | 0 \rangle$$

The normal ordering of any operator does not contribute anything in vacuum matrix element, because all the occupation numbers are zero in the vacuum.

$$= \langle 0 | \mathscr{N}(\phi_1 \phi_2) | 0 \rangle + \langle 0 | D_\mathrm{F}(\boldsymbol{x} - \boldsymbol{y}) | 0 \rangle$$

We use the linearity of the braket.

$$= \langle 0 | \mathscr{N}(\phi_1 \phi_2) + D_\mathrm{F}(\boldsymbol{x} - \boldsymbol{y}) | 0 \rangle$$

The contraction is defined via the Feynman propagator. We use this the other way around and insert a contraction.

$$= \left\langle 0 \middle| \mathscr{N}(\phi_1 \phi_2) + \wick{\c\phi_1 \c\phi_2} \middle| 0 \right\rangle$$

And as a last step we extend the normal ordering to the scalar.

$$= \left\langle 0 \middle| \mathscr{N}\left(\phi_1 \phi_2 + \wick{\c\phi_1 \c\phi_2}\right) \middle| 0 \right\rangle$$

Dropping the matrix element again, we can conclude that Equation (11) holds:

$$\mathcal{T}(\phi_1\phi_2) = \mathcal{N}\left(\phi_1\phi_2 + \contraction{}{\phi}{_1}{\phi}\phi_1\phi_2\right)$$

2.2 Proof of Wick's theorem

We saw in the previous subsection that the vacuum matrix elements are needed to get the propagators, but can be omitted, as long as we keep the vacuum in mind.

The theorem is proven for $n = 2$ fields. The case $n = 1$ is not very interesting since there is not much to reorder if there is just one field. The foundation for the induction is laid, we need the induction step next. We chose the variant to start with the left side at $n+1$, use the theorem at n and bring the other side to $n+1$ as well. The first step is extract the newest element from the time ordering.

Peskin and Schroeder (1995, p. 90) define the numbering of the fields such that $x_k^0 \geq x_{k+1}^0$, such that they are in time order already. If we do not use this trick as well, we would have $n+1$ terms when we write down the time ordering using Heaviside step functions explicitly. Therefore we will also use this trick. This means that ϕ_{n+1} has the smallest time of all and already is in the correct time order when it is at the very end. This does not mean that we loose generality by this restriction. We can always relabel the fields such that this holds: The time ordering $\mathcal{T}$ will order them by time anyway. And the normal ordering $\mathcal{N}$ splits up positive and negative frequency parts which can commute among themselves freely. All possible contractions are build up, so reordering the fields would not change anything either.

$$\mathcal{T}(\phi_1\phi_2\dots\phi_n\phi_{n+1}) = \mathcal{T}(\phi_1\phi_2\dots\phi_n)\phi_{n+1}$$

Then we can apply the theorem for n. To make the notation a tad more precise we call the sum of all contractions up to and including the field n the sum C_n.

$$= \mathcal{N}(\phi_1\phi_2\dots\phi_n + C_n)\phi_{n+1}$$

Here we need to split up the positive and negative frequency parts.

$$= \mathcal{N}(\phi_1\phi_2\dots\phi_n + C_n)\phi_{n+1}^+ + \mathcal{N}(\phi_1\phi_2\dots\phi_n + C_n)\phi_{n+1}^-$$

The normal ordering has the negative frequency parts on the left. This means that the positive frequency part already is in the right position and we can just insert that. The negative frequency part is on the wrong side to be normal ordered. We therefore need to commute it with every single positive frequency part. Every commutation will give us a commutator, which is a Feynman propagator when sandwiched into a vacuum matrix element.

$$\begin{aligned}&= \mathcal{N}(\phi_1\phi_2\dots\phi_n\phi_{n+1}^+ + C_n\phi_{n+1}^+) + \mathcal{N}(\phi_{n+1}^-\phi_1\phi_2\dots\phi_n + \phi_{n+1}^-C_n)\\ &\quad + \left[\mathcal{N}(\phi_1\phi_2\dots\phi_n), \phi_{n+1}^-\right] + \left[\mathcal{N}(C_n), \phi_{n+1}^-\right]\end{aligned}$$

The first summands of the terms in the first line can be combined to give the normal ordering of all $n+1$ fields. The terms with C_n will give all the contractions where ϕ_{n+1} is not involved. The first commutator will then give all the contractions that ϕ_{n+1} is involved in, but nothing else is contracted. Finally, the commutator of ϕ_{n+1} with C_n will give all the contractions where ϕ_{n+1} is involved and other things are contracted as well. All in all we arrive at the final result:

$$= \mathcal{N}(\phi_1 \phi_2 \dots \phi_n \phi_{n+1}).$$

By the method of induction, this theorem holds for all $n \geq 1$ now.

References

Peskin, Michael E. and Daniel V. Schroeder (1995). *An Introduction to Quantum Field Theory*. Westview Press. ISBN: 978-0-201-50397-5.