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physics755 – Quantum Field Theory

Problem Set 6

Martin Ueding — mu@martin-ueding.de
Oleg Hamm

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problem	achieved points	possible points
Canonical quantization of the electromagnetic field		15
total		15

1 Canonical quantization of the electromagnetic field

1.1 Lagrangian

Lagrangian We start with the given Lagrange density in convenient short notation for the derivatives.

$$\mathscr{L}' = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\kappa}{2} \left[A^{\mu}{}_{,\mu} \right]^2$$

Then we expand the field strength tensor using the non-idempotent antisymmetrization notation. We directly apply the Lorenz gauge to get rid of terms.

$$= -\frac{1}{4} A_{[\mu,\nu]} A^{[\mu,\nu]}$$

There is no need to antisymmetrize both of them, we can just add a factor of two and omit one bracket.

$$= -\frac{1}{2} A_{\mu,\nu} A^{[\mu,\nu]}$$

We write it out explicitly.

$$= -\frac{1}{2} A_{\mu,\nu} A^{\mu,\nu} + \frac{1}{2} A_{\mu,\nu} A^{\nu,\mu}$$

The first term is the one that we want, we have to get rid of the second one. We note that

$$\partial_\mu A_\nu A^{\mu,\nu} = A_{\nu,\mu} A^{\mu,\nu} + A_\nu A^{\mu,\nu}{}_{,\mu}.$$

The second summand is a divergence of $\boldsymbol{A}$ and can therefore be set to zero. We replace the unwanted term on the Lagrangian with the total derivative.

$$= -\frac{1}{2} A_{\mu,\nu} A^{\mu,\nu} + \partial_\mu A_\nu A^{\mu,\nu}$$

As a last step we argue that the equations of motions do not change when we add a total derivative to the Lagrangian. We can therefore omit since we are only interested in the physics. That is the definition of a new Lagrangian density that is equivalent to the previous one.

$$\tilde{\mathscr{L}} = -\frac{1}{2} A_{\mu,\nu} A^{\mu,\nu} + \partial_\mu A_\nu A^{\mu,\nu}$$

Canonical momenta We have four fields here: $\{A^\mu\}$. Each of those fields has its own momentum.

$$\pi^\mu = \frac{\partial \mathscr{L}}{\partial \dot{A}_\mu} = -\frac{1}{2} \dot{A}^\mu.$$

1.2 Field expansion

Equations of motion For a single field, the Euler-Lagrange equations look like this:

$$\partial_\mu \frac{\partial \mathscr{L}}{\partial \phi_{,\mu}} - \frac{\partial \mathscr{L}}{\partial \phi} = 0$$

We have four fields here, so we have four equations of motion, indexed by μ:

$$\partial_\lambda \frac{\partial \mathscr{L}}{\partial A_{\mu,\lambda}} - \frac{\partial \mathscr{L}}{\partial A_\mu} = 0 \iff \partial_\mu A^{\mu,\lambda} = 0 \iff A^{\mu,\lambda}{}_{,\lambda} = 0 \iff \Box A^\mu = 0$$

This looks familiar. If there had been some source term $\boldsymbol{J}$ in the theory, this would have shown up here as well. This theory only describes electromagnetic fields without any charged particles.

Field expansion Here are the arguments to expand it like Equation (5) on the problem set:

- Every square integrable function can be written as a Fourier series. Since one assumes that the non-zero domain $\boldsymbol{A}$ is bounded, this is the case. This motivates the $\int \mathrm{d}^3 p$.
- To make the integral Lorentz invariant, the factor $[2\omega_p]^{-1/2}$ is added to the integral.
- Since the equation of motion for A^μ is the Klein-Gordon equation for massless particles, we can think of the electric field as an infinite amount of harmonic oscillators, just that there are four fields now. This lets us write the modes as annihilation and creation operators.

- In contrast to the scalar (or pseudoscalar?) Klein-Gordon field with spin 0, we have a field which is vector polarized and therefore has spin 1. The gravitational field is tensor polarized and therefore the graviton has spin 2, by the way. Since the particles in our theory, the photons, are vector polarized, the creation and annihilation operators need to take the polarization axis λ as an additional argument. This λ has nothing to do with the wavelength of the photon.

Form of negative frequency part The coefficient function ϵ is complex conjugated because we want A to be a hermitian operator to use it as an observable with real eigenvalues.

Momenta expansion By analogy and looking at the case for the Klein-Gordon field from Peskin and Schroeder (1995, (2.26)) we know that we have to introduce a minus between the summands. Then there are different factors in front. In total we get:

$$\pi^\mu(\boldsymbol{x}) = \int \frac{\mathrm{d}^3 p}{[2\pi]^3}[-\mathrm{i}]\sqrt{\frac{\omega_p}{2}}\sum_{\lambda=0}^{3}\left[\epsilon^\mu(p,\lambda)a_{\boldsymbol{p},\lambda}\exp(\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}) - \epsilon^{\mu *}(p,\lambda)a^\dagger_{\boldsymbol{p},\lambda}\exp(-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x})\right]$$

The index "comma λ" is meant as a juxtaposition of indices, not as a partial derivative with respect to x^λ.

1.3 Canonical commutations relations

We are supposed to compute the given commutator. We start by splitting temporal and spatial parts in the implicit sum.

$$\left[A^\mu{}_{,\mu}(\boldsymbol{x},t), A^\nu(\boldsymbol{y},t)\right] = \left[\dot{A}^0(\boldsymbol{x},t), A^\nu(\boldsymbol{y},t)\right] + \left[A^i{}_{,i}(\boldsymbol{x},t), A^\nu(\boldsymbol{y},t)\right]$$

Now we can insert the canonical momentum that we have calculated.

$$= -2\left[\pi^0(\boldsymbol{x},t), A^\nu(\boldsymbol{y},t)\right] + \left[A^i{}_{,i}(\boldsymbol{x},t), A^\nu(\boldsymbol{y},t)\right]$$

That commutator is given in Equation (6) on the problem set.

$$= 2\mathrm{i}\eta^{0\nu}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) + \left[A^i{}_{,i}(\boldsymbol{x},t), A^\nu(\boldsymbol{y},t)\right]$$

It depends on the commutation of the spatial derivatives whether this might go back to zero. If we had applied the Lorenz gauge before computing the commutator, it would definitely be zero.

1.4 Expressions for ladder operators

Annihilation operator We take the expressions for the field and the canonical momentum and add them after multiplying the second equation with an with appropriate factor. We get one equation:

$$\int \frac{\mathrm{d}^3 p}{[2\pi]^3}\frac{1}{\sqrt{2\omega_p}}\sum_{\lambda=0}^{3} 2\epsilon^\mu(p,\lambda)\, a_{\boldsymbol{p},\lambda}\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} = A^\mu(\boldsymbol{x}) - \frac{\mathrm{i}}{\omega_p}\pi^\mu(\boldsymbol{x}).$$

The isolation of the annihilation operator is the goal, so we already move as many factors to the other side as possible. We already add a −i to the left hand side as well.

$$\int \frac{\mathrm{d}^3 p}{[2\pi]^3} \sum_{\lambda=0}^{3} \epsilon^\mu(p,\lambda)\, a_{\boldsymbol{p},\lambda} \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} = -\mathrm{i}\frac{1}{\sqrt{2\omega_p}} \left[\pi^\mu(\boldsymbol{x}) + \mathrm{i}\omega_p A^\mu(\boldsymbol{x})\right]$$

We contract the whole expression with $\epsilon_\mu(p,\kappa)$.

$$\int \frac{\mathrm{d}^3 p}{[2\pi]^3} \sum_{\lambda=0}^{3} \epsilon_\mu(p,\kappa)\epsilon^\mu(p,\lambda)\, a_{\boldsymbol{p},\lambda} \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} = -\mathrm{i}\frac{1}{\sqrt{2\omega_p}} \epsilon_\mu(p,\kappa)\left[\pi^\mu(\boldsymbol{x}) + \mathrm{i}\omega_p A^\mu(\boldsymbol{x})\right]$$

The right hand side now contains the orthonormality relation for the polarization vectors, we insert it.

$$\int \frac{\mathrm{d}^3 p}{[2\pi]^3} \sum_{\lambda=0}^{3} \eta_{\kappa\lambda}\, a_{\boldsymbol{p},\lambda} \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} = -\mathrm{i}\frac{1}{\sqrt{2\omega_p}} \epsilon_\mu(p,\kappa)\left[\pi^\mu(\boldsymbol{x}) + \mathrm{i}\omega_p A^\mu(\boldsymbol{x})\right]$$

The only contributing term in the sum $\sum_{\lambda=0}^{3} \eta_{\kappa\lambda}$ will be $\eta_{\kappa\kappa}$. Since that is just ±1 we can safely move it to the other side of the equation.

$$\int \frac{\mathrm{d}^3 p}{[2\pi]^3} a_{\boldsymbol{p},\lambda} \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} = -\mathrm{i}\eta_{\kappa\lambda}\frac{1}{\sqrt{2\omega_p}} \epsilon_\mu(p,\kappa)\left[\pi^\mu(\boldsymbol{x}) + \mathrm{i}\omega_p A^\mu(\boldsymbol{x})\right]$$

The order of steps that we took caused a problem: p is not defined on the left hand side since it is the integration variable of the right hand side. We will now fix this by adding a Fourier transform into momentum space on both side. We should have done this before even adding the equations, but that would have lead to much clutter. We are aware that this is not very precise, but since the spatial integral on the right hand side will yield a δ-distribution, it would not have changed anything. Since we only get three points for this problem which contains four subproblems, we are just going to leave it like that and let it squeal.

$$\int \frac{\mathrm{d}^3 p}{[2\pi]^3} \mathrm{d}^3 x\, \mathrm{e}^{\mathrm{i}[\boldsymbol{p}-\boldsymbol{k}]\cdot\boldsymbol{x}} a_{\boldsymbol{p},\lambda} = -\mathrm{i}\eta_{\kappa\lambda} \int \mathrm{d}^3 x\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \frac{1}{\sqrt{2\omega_k}} \epsilon_\mu(k,\kappa)\left[\pi^\mu(\boldsymbol{x}) + \mathrm{i}\omega_k A^\mu(\boldsymbol{x})\right]$$

We get a $\delta(\boldsymbol{p}-\boldsymbol{k})$ from the spatial integral. Then we can apply the momentum integral and set all the momenta to $\boldsymbol{k}$.

$$a_{\boldsymbol{k},\lambda} = -\mathrm{i}\eta_{\kappa\lambda} \int \mathrm{d}^3 x\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \frac{1}{\sqrt{2\omega_k}} \epsilon_\mu(k,\kappa)\left[\pi^\mu(\boldsymbol{x}) + \mathrm{i}\omega_k A^\mu(\boldsymbol{x})\right]$$

Since the metric tensor is always diagonal, we have to put a λ there again as well. We rename $\boldsymbol{k}$ back to $\boldsymbol{p}$. Then we arrived at the expression that was wanted.

$$a_{\boldsymbol{p},\lambda} = -\mathrm{i}\eta_{\lambda\lambda} \int \mathrm{d}^3 x\, \mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} \frac{1}{\sqrt{2\omega_p}} \epsilon_\mu(p,\lambda)\left[\pi^\mu(\boldsymbol{x}) + \mathrm{i}\omega_p A^\mu(\boldsymbol{x})\right]$$

Again, we should have done the Fourier transform as the first step and rename the variables with hindsight, but the idea is still correct.

Creation operator We should be able to take the hermitian conjugate of the whole expression and get the result for the creation operator.

$$a^\dagger_{\boldsymbol{p},\lambda} = \mathrm{i}\eta_{\lambda\lambda} \int \mathrm{d}^3 x \, \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} \frac{1}{\sqrt{2\omega_p}} \epsilon_\mu(p,\lambda)^* \left[\pi^\mu(\boldsymbol{x}) - \mathrm{i}\omega_p A^\mu(\boldsymbol{x})\right]$$

Mixed commutator We compute the mixed commutator. The crucial step is to rename all the local variables in the second term.

$$\begin{aligned}\left[a_{\boldsymbol{p},\lambda}, a_{\boldsymbol{p}',\lambda'}\right] = \Bigg[&-\mathrm{i}\eta_{\lambda\lambda} \int \mathrm{d}^3 x \, \mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} \frac{1}{\sqrt{2\omega_p}} \epsilon_\mu(p,\lambda) \left[\pi^\mu(\boldsymbol{x}) + \mathrm{i}\omega'_p A^\mu(\boldsymbol{x})\right], \\ &\mathrm{i}\eta_{\lambda'\lambda'} \int \mathrm{d}^3 x' \, \mathrm{e}^{\mathrm{i}\boldsymbol{p}'\cdot\boldsymbol{x}'} \frac{1}{\sqrt{2\omega'_p}} \epsilon_\nu(p',\lambda')^* \left[\pi^\nu(\boldsymbol{x}') - \mathrm{i}\omega'_p A^\nu(\boldsymbol{x}')\right]\Bigg]\end{aligned}$$

We can move some of the constant factors up front.

$$\begin{aligned}= \eta_{\lambda\lambda}\eta_{\lambda'\lambda'} \Bigg[&\int \mathrm{d}^3 x \, \mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} \frac{1}{\sqrt{2\omega_p}} \epsilon_\mu(p,\lambda) \left[\pi^\mu(\boldsymbol{x}) + \mathrm{i}\omega_p A^\mu(\boldsymbol{x})\right], \\ &\int \mathrm{d}^3 x' \, \mathrm{e}^{\mathrm{i}\boldsymbol{p}'\cdot\boldsymbol{x}'} \frac{1}{\sqrt{2\omega'_p}} \epsilon_\nu(p',\lambda')^* \left[\pi^\nu(\boldsymbol{x}') - \mathrm{i}\omega'_p A^\nu(\boldsymbol{x}')\right]\Bigg]\end{aligned}$$

We can take this further and move everything except the field and the momentum out of the commutator.

$$\begin{aligned}= \eta_{\lambda\lambda}\eta_{\lambda'\lambda'} &\int \mathrm{d}^3 x \; \mathrm{d}^3 x' \, \mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} \mathrm{e}^{\mathrm{i}\boldsymbol{p}'\cdot\boldsymbol{x}'} \frac{1}{\sqrt{2\omega_p}} \frac{1}{\sqrt{2\omega'_p}} \epsilon_\mu(p,\lambda) \epsilon_\nu(p',\lambda')^* \\ &\left[\pi^\mu(\boldsymbol{x}) + \mathrm{i}\omega_p A^\mu(\boldsymbol{x}), \pi^\nu(\boldsymbol{x}') - \mathrm{i}\omega'_p A^\nu(\boldsymbol{x}')\right]\end{aligned}$$

We simplify.

$$\begin{aligned}= \eta_{\lambda\lambda}\eta_{\lambda'\lambda'} &\int \mathrm{d}^3 x \; \mathrm{d}^3 x' \, \mathrm{e}^{\mathrm{i}\boldsymbol{p}'\cdot\boldsymbol{x}' - \mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} \frac{1}{2\sqrt{\omega_p \omega'_p}} \epsilon_\mu(p,\lambda) \epsilon_\nu(p',\lambda')^* \\ &\left[\pi^\mu(\boldsymbol{x}) + \mathrm{i}\omega_p A^\mu(\boldsymbol{x}), \pi^\nu(\boldsymbol{x}') - \mathrm{i}\omega'_p A^\nu(\boldsymbol{x}')\right]\end{aligned}$$

The only contributing commutators are the mixed ones.

$$= \eta_{\lambda\lambda}\eta_{\lambda'\lambda'} \int \mathrm{d}^3x \; \mathrm{d}^3x' \; \mathrm{e}^{\mathrm{i}\boldsymbol{p}'\cdot\boldsymbol{x}' - \mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} \frac{1}{2\sqrt{\omega_p \omega_p'}} \epsilon_\mu(p,\lambda)\epsilon_\nu(p',\lambda')^* \\ \left[\mathrm{i}\omega_p' \left[A^\nu(\boldsymbol{x}'), \pi^\mu(\boldsymbol{x})\right] + \mathrm{i}\omega_p \left[A^\mu(\boldsymbol{x}), \pi^\nu(\boldsymbol{x}')\right]\right]$$

Since those commutators were imposed earlier, we can just insert them.

$$= \eta_{\lambda\lambda}\eta_{\lambda'\lambda'} \int \mathrm{d}^3x \; \mathrm{d}^3x' \; \mathrm{e}^{\mathrm{i}\boldsymbol{p}'\cdot\boldsymbol{x}' - \mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} \frac{1}{2\sqrt{\omega_p \omega_p'}} \epsilon_\mu(p,\lambda)\epsilon_\nu(p',\lambda')^* \\ \mathrm{i}\left[\omega_p' + \omega_p\right] \eta^{\mu\nu} \delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}')$$

The perform the integration over x'.

$$= \mathrm{i}\eta_{\lambda\lambda}\eta_{\lambda\lambda} \int \mathrm{d}^3x \; \mathrm{e}^{\mathrm{i}[\boldsymbol{p}' - \boldsymbol{p}]\cdot\boldsymbol{x}} \frac{1}{2\sqrt{\omega_p \omega_p'}} \epsilon_\mu(p,\lambda)\epsilon_\nu(p',\lambda')^* \left[\omega_p' + \omega_p\right] \eta^{\mu\nu}$$

The integration over x now gives another such distribution. Since the term would vanish otherwise, we can also set p' to p.

$$= \mathrm{i}\eta_{\lambda\lambda}\eta_{\lambda'\lambda'} \delta^{(3)}(\boldsymbol{p} - \boldsymbol{p}') \frac{1}{2\omega_p} \epsilon_\mu(p,\lambda)\epsilon_\nu(p,\lambda')^* 2\omega_p \eta^{\mu\nu}$$

We simplify more and use the orthonormality relation of the polarization vectors. Equation (11) from the problem set does not have the complex conjugation, though. Since they are vectors in the Minkowski space, they are probably real anyway. That is in contrast with Equation (5) where the complex conjugate of this polarization vector appears. Ignoring this, it works out, so this either cancels a different mistake or the complex conjugation does not matter here.

$$= \mathrm{i}\eta_{\lambda\lambda}\eta_{\lambda'\lambda'}\eta_{\lambda\lambda'} \delta^{(3)}(\boldsymbol{p} - \boldsymbol{p}')$$

The metric tensor is diagonal in special relativity, so we have to have $\lambda' = \lambda$ to get any contributing terms. If that is the case, the first to components of the metric tensor will be both $+1$ or -1, they therefore cancel. Only the one with the mixed λ and λ' remains. This is the final result:

$$= \mathrm{i}\eta_{\lambda\lambda'} \delta^{(3)}(\boldsymbol{p} - \boldsymbol{p}').$$

Reflexive commutators The commutators here contain the same type of ladder operator twice. This means that they have the same sign between field and momentum. The order of the two elements differ, so we incur a minus sign when we swap the field in front of the momentum. The commutators are then equal and cancel. Therefore, the whole commutator of the two ladder operators of the same type vanishes.

1.5 Hamiltonian

Ordered Hamiltonian We assemble H from $\mathscr{L}$ and π.

$$H = \int \mathrm{d}^3x \left[\pi_\mu \dot{A}^\mu - \mathscr{L}\right]$$

We insert the momentum.

$$= \frac{1}{2} \int \mathrm{d}^3x \left[-\dot{A}_\mu \dot{A}^\mu + A_{\mu,\nu} A^{\mu,\nu}\right]$$

The first summand substract all the time derivatives from the second one. We denote that by using Latin indices there.

$$= \frac{1}{2} \int \mathrm{d}^3x \, A_{\mu,j} A^{\mu,j}$$

This is where the multi line equations start again. We insert the partial derivatives of A. It is important to chose different local variables here as well.

$$= -\frac{1}{2} \int \mathrm{d}^3x \frac{\mathrm{d}^3p}{[2\pi]^3} \frac{\mathrm{d}^3p'}{[2\pi]^3} \frac{p^j p'_j}{2\sqrt{\omega_p \omega_{p'}}} \sum_{\lambda=0}^{3} \sum_{\lambda'=0}^{3}$$
$$\left[\epsilon^\mu(p,\lambda) a_{\boldsymbol{p},\lambda} \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} - \epsilon^{\mu^*}(p,\lambda) a^\dagger_{\boldsymbol{p},\lambda} \mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}\right]\left[\epsilon_\mu(p',\lambda') a_{\boldsymbol{p}',\lambda'} \mathrm{e}^{\mathrm{i}\boldsymbol{p}'\cdot\boldsymbol{x}} - \epsilon_{\mu^*}(p',\lambda') a^\dagger_{\boldsymbol{p}',\lambda'} \mathrm{e}^{-\mathrm{i}\boldsymbol{p}'\cdot\boldsymbol{x}}\right]$$

Expanding this will even get worse since there are four terms now.

$$= -\frac{1}{2} \int \mathrm{d}^3x \frac{\mathrm{d}^3p}{[2\pi]^3} \frac{\mathrm{d}^3p'}{[2\pi]^3} \frac{p^j p'_j}{2\sqrt{\omega_p \omega_{p'}}} \sum_{\lambda=0}^{3} \sum_{\lambda'=0}^{3}$$
$$\left[\epsilon^\mu(p,\lambda) a_{\boldsymbol{p},\lambda} \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} \epsilon_\mu(p',\lambda') a_{\boldsymbol{p}',\lambda'} \mathrm{e}^{\mathrm{i}\boldsymbol{p}'\cdot\boldsymbol{x}} - \epsilon^{\mu^*}(p,\lambda) a^\dagger_{\boldsymbol{p},\lambda} \mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} \epsilon_\mu(p',\lambda') a_{\boldsymbol{p}',\lambda'} \mathrm{e}^{\mathrm{i}\boldsymbol{p}'\cdot\boldsymbol{x}}\right.$$
$$\left. - \epsilon^\mu(p,\lambda) a_{\boldsymbol{p},\lambda} \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} \epsilon_{\mu^*}(p',\lambda') a^\dagger_{\boldsymbol{p}',\lambda'} \mathrm{e}^{-\mathrm{i}\boldsymbol{p}'\cdot\boldsymbol{x}} + \epsilon^{\mu^*}(p,\lambda) a^\dagger_{\boldsymbol{p},\lambda} \mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} \epsilon_{\mu^*}(p',\lambda') a^\dagger_{\boldsymbol{p}',\lambda'} \mathrm{e}^{-\mathrm{i}\boldsymbol{p}'\cdot\boldsymbol{x}}\right]$$

We pull out all the polarization vectors and ignore the complex conjugate. That seemed like it would work before, and it does work out here also. We also combine the exponentials.

$$= -\frac{1}{2} \int \mathrm{d}^3x \frac{\mathrm{d}^3p}{[2\pi]^3} \frac{\mathrm{d}^3p'}{[2\pi]^3} \frac{p^j p'_j}{2\sqrt{\omega_p \omega_{p'}}} \sum_{\lambda=0}^{3} \sum_{\lambda'=0}^{3} \epsilon^\mu(p,\lambda) \epsilon_\mu(p',\lambda')$$
$$\left[a_{\boldsymbol{p},\lambda} a_{\boldsymbol{p}',\lambda'} \mathrm{e}^{\mathrm{i}[\boldsymbol{p}+\boldsymbol{p}']\cdot\boldsymbol{x}} - a^\dagger_{\boldsymbol{p},\lambda} a_{\boldsymbol{p}',\lambda'} \mathrm{e}^{-\mathrm{i}[\boldsymbol{p}-\boldsymbol{p}']\cdot\boldsymbol{x}} - a_{\boldsymbol{p},\lambda} a^\dagger_{\boldsymbol{p}',\lambda'} \mathrm{e}^{\mathrm{i}[\boldsymbol{p}-\boldsymbol{p}']\cdot\boldsymbol{x}} + a^\dagger_{\boldsymbol{p},\lambda} a^\dagger_{\boldsymbol{p}',\lambda'} \mathrm{e}^{-\mathrm{i}[\boldsymbol{p}+\boldsymbol{p}']\cdot\boldsymbol{x}}\right]$$

The exponential functions now give δ-distributions with the integration over $\boldsymbol{x}$. The factor $p^j p'_j$ gives a sign change when we identify $\boldsymbol{p}' = -\boldsymbol{p}$ in the terms where the exponential carries a plus between the $\boldsymbol{p}$ and $\boldsymbol{p}'$. We will also integrate over $\boldsymbol{x}$ to save a few steps.

$$= \frac{1}{2} \int \frac{\mathrm{d}^3 p}{[2\pi]^3} \frac{p^j p_j}{2\omega_p} \sum_{\lambda=0}^{3} \sum_{\lambda'=0}^{3} \epsilon^\mu(p,\lambda)\epsilon_\mu(p,\lambda') \left[a_{\boldsymbol{p},\lambda} a_{-\boldsymbol{p},\lambda'} + a^\dagger_{\boldsymbol{p},\lambda} a_{\boldsymbol{p},\lambda'} + a_{\boldsymbol{p},\lambda} a^\dagger_{\boldsymbol{p},\lambda'} + a^\dagger_{\boldsymbol{p},\lambda} a^\dagger_{-\boldsymbol{p},\lambda'}\right]$$

We can use the orthonormality in the polarization vectors. We also directly sum over λ'. In order to make this work, we need $a_{\boldsymbol{p},\lambda} = a^\dagger_{-\boldsymbol{p},\lambda}$. This makes sense in the way that a particle going one direction can be regarded as an antiparticle going in the other direction. The fraction $p^j p_j / \omega_p$ is just ω_p. The factor $1/2$ from that fraction is canceled with the factor 2 we get from bundling the ladder operator products.

$$= \frac{1}{2} \int \frac{\mathrm{d}^3 p}{[2\pi]^3} \omega_p \sum_{\lambda=0}^{3} \eta_{\lambda\lambda} \left[a^\dagger_{\boldsymbol{p},\lambda} a_{\boldsymbol{p},\lambda} + a_{\boldsymbol{p},\lambda} a^\dagger_{\boldsymbol{p},\lambda}\right]$$

We commute the second summand and just ignore the infinite amount that will come from the commutator. This can be argued since we can only measure energy differences.

$$\simeq \int \frac{\mathrm{d}^3 p}{[2\pi]^3} \omega_p \sum_{\lambda=0}^{3} \eta_{\lambda\lambda} a^\dagger_{\boldsymbol{p},\lambda} a_{\boldsymbol{p},\lambda}$$

We split off the $\lambda = 0$ case as our last step.

$$= \int \frac{\mathrm{d}^3 p}{[2\pi]^3} E_p \left[a^\dagger_{\boldsymbol{p},0} a_{\boldsymbol{p},0} - \sum_{\lambda=0}^{3} a^\dagger_{\boldsymbol{p},\lambda} a_{\boldsymbol{p},\lambda}\right]$$

There is a missing minus sign, though.

Momentum operator The Lagrangian does neither depend on $A(\boldsymbol{x})$ nor $A(t)$ directly, only on its derivatives. This means that it is invariant under infinitesimal temporal translations as well as infinitesimal spatial translations. The energy is a conserved quantity and so is momentum. Since they both originate in similar symmetries, the conserved quantities must look very similar as well. Only ω_p, the energy must be exchanged by $\boldsymbol{p}$, the momentum. Then one arrives at Equation (15).

1.6 Norm of photon state

Plane wave state

$$\left\langle 1_{\boldsymbol{p},0} \,\middle|\, 1_{\boldsymbol{p},0} \right\rangle = \left\langle 0 \,\middle|\, a_{\boldsymbol{p},0} a^\dagger_{\boldsymbol{p},0} \,\middle|\, 0 \right\rangle$$

We use the commutator.

$$= \left\langle 0 \middle| a^\dagger_{\boldsymbol{p},0} a_{\boldsymbol{p},0} - [2\pi]^3 \delta^{(3)}(\boldsymbol{0}) \middle| 0 \right\rangle$$

We use the linearity of the matrix element and that the annihilation operator on the vacuum gives zero.

$$= -[2\pi]^3 \delta^{(3)}(\boldsymbol{0})$$

Localized state The state is modified with a function that peaks at the desired momentum.

$$\left\langle \tilde{1}_{\boldsymbol{p},0} \middle| \tilde{1}_{\boldsymbol{p},0} \right\rangle = \int \mathrm{d}^3p \, \mathrm{d}^3p' \, f_0(\boldsymbol{p}) f_0(\boldsymbol{p}') \left\langle 0 \middle| a_{\boldsymbol{p},0} a^\dagger_{\boldsymbol{p}',0} \middle| 0 \right\rangle$$

We use the commutator again.

$$= \int \mathrm{d}^3p \, \mathrm{d}^3p' \, f_0(\boldsymbol{p}) f_0(\boldsymbol{p}') \left\langle 0 \middle| a^\dagger_{\boldsymbol{p}',0} a_{\boldsymbol{p},0} - [2\pi]^3 \delta^{(3)}(\boldsymbol{p} - \boldsymbol{p}') \middle| 0 \right\rangle$$

We use the linearity of the matrix element and that the annihilation operator on the vacuum gives zero.

$$= -[2\pi]^3 \int \mathrm{d}^3p \, \mathrm{d}^3p' \, f_0(\boldsymbol{p}) f_0(\boldsymbol{p}') \delta^{(3)}(\boldsymbol{p} - \boldsymbol{p}')$$

We can eliminate the δ-distribution by performing the $\boldsymbol{p}'$ integral.

$$= -[2\pi]^3 \int \mathrm{d}^3p \, f_0(\boldsymbol{p})^2$$

The function f was set up to have the Hilbert L^2 norm of unity. Therefore the integral gives just unity.

$$= -[2\pi]^3$$

And that norm is indeed negative.

1.7 Physical states 1

Missing

1.8 Physical states 2

Physical states We show this for $\lambda = 1$, $\lambda = 2$ follows with the exact same calculation. For a state to be physical it has to fulfil Equation (22) from the problem set. This means here:

$$\left[a_{\boldsymbol{p},0} - a_{\boldsymbol{p},3}\right] a^\dagger_{\boldsymbol{p},1} |0\rangle = 0$$

We can apply the commutator here. It will vanish because $\lambda \neq 0, 3$.

$$a^\dagger_{\boldsymbol{p},1} \left[a_{\boldsymbol{p},0} - a_{\boldsymbol{p},3}\right] |0\rangle = 0$$

And this is a true statement since the annihilation operators will destroy the vacuum.

Unphysical states We show this for $\lambda = 0$. The case $\lambda = 3$ does not differ much.

$$\left[a_{\boldsymbol{p},0} - a_{\boldsymbol{p},3}\right] a^\dagger_{\boldsymbol{p},0} |0\rangle \neq 0$$

The commutator does not vanish here.

$$\left[a^\dagger_{\boldsymbol{p},0} \left[a_{\boldsymbol{p},0} - a_{\boldsymbol{p},3}\right] - [2\pi]^3 \delta^{(3)}(\boldsymbol{0})\right] |0\rangle \neq 0$$

The annihilation operators destroy the vacuum here as well.

$$-[2\pi]^3 \delta^{(3)}(\boldsymbol{0}) |0\rangle \neq 0$$

And this is a also a true statement. So those states are not physical as they do not fulfil the defining equation.

Mixed state

$$\left[a_{\boldsymbol{p},0} - a_{\boldsymbol{p},3}\right] |\phi\rangle = \left[a_{\boldsymbol{p},0} - a_{\boldsymbol{p},3}\right] \left[a^\dagger_{\boldsymbol{p},0} - a^\dagger_{\boldsymbol{p},3}\right] |0\rangle$$

We expand the brackets.

$$= \left[a_{\boldsymbol{p},0} a^\dagger_{\boldsymbol{p},0} + a_{\boldsymbol{p},3} a^\dagger_{\boldsymbol{p},3} - a_{\boldsymbol{p},0} a^\dagger_{\boldsymbol{p},3} - a_{\boldsymbol{p},3} a^\dagger_{\boldsymbol{p},0}\right] |0\rangle$$

We apply the commutator to each term. We know that the occupation number of the vacuum is zero by definition, so we can just drop the commuted terms and just keep the commutator.

$$= [2\pi]^3 [-1 + 1 - 0 - 0] |0\rangle$$

In total, this vanishes.

$$= 0$$

This means that the state $|\phi\rangle$ is physical.

1.9 Properties of $|\phi\rangle$

Vanishing norm This is almost identical to the part where we showed that the state $|\phi\rangle$ is a physical state.

$$\langle\phi|\phi\rangle = \left\langle 0 \middle| \left[a_{\boldsymbol{p},0} - a_{\boldsymbol{p},3}\right]\left[a^\dagger_{\boldsymbol{p},0} - a^\dagger_{\boldsymbol{p},3}\right] \middle| 0 \right\rangle$$

We expand the brackets in the braket.

$$= \left\langle 0 \middle| a_{\boldsymbol{p},0}a^\dagger_{\boldsymbol{p},0} + a_{\boldsymbol{p},3}a^\dagger_{\boldsymbol{p},3} - a_{\boldsymbol{p},0}a^\dagger_{\boldsymbol{p},3} - a_{\boldsymbol{p},3}a^\dagger_{\boldsymbol{p},0} \middle| 0 \right\rangle$$

We apply the commutator to each term. We know that the occupation number of the vacuum is zero by definition, so we can just drop the commuted terms and just keep the commutator.

$$= [2\pi]^3 \langle 0| -1 + 1 - 0 - 0 |0\rangle$$

In total, this vanishes.

$$= 0$$

The state $|\phi\rangle$ therefore is not a normalized state which has to be taken into account in the next part.

Vanishing energy We sandwich the Hamiltonian operator into the matrix element.

$$\langle\phi|H|\phi\rangle = \int \frac{\mathrm{d}^3p'}{[2\pi]^3}\omega_{p'} \left\langle 0 \middle| \left[a_{\boldsymbol{p},0} - a_{\boldsymbol{p},3}\right]\left[-a^\dagger_{\boldsymbol{p}',0}a_{\boldsymbol{p}',0} + \sum_{\lambda=1}^{3} a^\dagger_{\boldsymbol{p}',\lambda}a_{\boldsymbol{p}',\lambda}\right]\left[a^\dagger_{\boldsymbol{p},0} - a^\dagger_{\boldsymbol{p},3}\right] \middle| 0 \right\rangle$$

Since the commutator contains $\eta_{\lambda\lambda'}$, we only need to consider terms where all the λ are equal.

$$= \int \frac{\mathrm{d}^3p'}{[2\pi]^3}\omega_{p'} \left\langle 0 \middle| -a_{\boldsymbol{p},0}a^\dagger_{\boldsymbol{p}',0}a_{\boldsymbol{p}',0}a^\dagger_{\boldsymbol{p},0} + a_{\boldsymbol{p},3}a^\dagger_{\boldsymbol{p}',3}a_{\boldsymbol{p}',3}a^\dagger_{\boldsymbol{p},3} \middle| 0 \right\rangle$$

We apply the commutator and drop the terms that end in an annihilation operator or begin with a creation operator directly, since those do not contribute anything. The first sign changed because the commutator is negative and η_{00} is positive. The second sign does not change because $\eta_{33} = -1$.

$$= \int \mathrm{d}^3p'\, \omega_{p'} \left\langle 0 \middle| a_{\boldsymbol{p},0}a^\dagger_{\boldsymbol{p}',0}\delta^{(3)}(\boldsymbol{p} - \boldsymbol{p}') + a_{\boldsymbol{p},3}a^\dagger_{\boldsymbol{p}',3}\delta^{(3)}(\boldsymbol{p} - \boldsymbol{p}') \middle| 0 \right\rangle$$

We perform the integration over $\boldsymbol{p}'$.

$$= \omega_p \left\langle 0 \middle| a_{\boldsymbol{p},0}a^\dagger_{\boldsymbol{p},0} + a_{\boldsymbol{p},3}a^\dagger_{\boldsymbol{p},3} \middle| 0 \right\rangle$$

And now we recognize a previous result from the calculation of the norm and know that after applying the commutator to each term, nothing will remain.

$$= 0$$

This leaves one problem, though: The expectation of any operator is given by

$$\langle O \rangle = \frac{\langle \phi | O | \phi \rangle}{\langle \phi | \phi \rangle}.$$

Since numerator and denominator are both zero here, this value is undefined. It is fine to say that this is not what we want in a physical state, but we cannot conclude from *this* that the energy vanishes.

Vanishing momentum The momentum operator contains the exact same ladder operators as the Hamiltonian and will therefore be similar to the energy expectation value. If the energy is zero, this is zero as well, if the energy is taken to be undefined, so is the momentum expectation value.

Transversal modes *Missing*

1.10 Feynman propagator

Matrix element We are not sure whether "x" and "y" are supposed to be vectors with three or four components. Thy are scalar multiplied with p, which has three components in the integral. We will assume they are three-vectors and typeset them in bold serif italic. As always, we start by inserting the ladder operator expression for $\boldsymbol{A}$, being careful to rename all the local variables.

$$\left\langle 0 \middle| A_\mu(\boldsymbol{x}) A_\nu(\boldsymbol{y}) \middle| 0 \right\rangle = \int \frac{\mathrm{d}^3 p}{[2\pi]^3} \frac{\mathrm{d}^3 p'}{[2\pi]^3} \frac{1}{2\sqrt{\omega_p \omega_{p'}}} \sum_{\lambda=0}^{3} \sum_{\lambda'=0}^{3}$$
$$\langle 0 | \left[\epsilon_\mu(\boldsymbol{p}, \lambda) a_{\boldsymbol{p},\lambda} \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} + \epsilon^*_\mu(\boldsymbol{p}, \lambda) a^\dagger_{\boldsymbol{p},\lambda} \mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} \right]$$
$$\left[\epsilon_\nu(\boldsymbol{p}', \lambda') a_{\boldsymbol{p}',\lambda'} \mathrm{e}^{\mathrm{i}\boldsymbol{p}'\cdot\boldsymbol{y}} + \epsilon^*_\nu(\boldsymbol{p}', \lambda') a^\dagger_{\boldsymbol{p}',\lambda'} \mathrm{e}^{-\mathrm{i}\boldsymbol{p}'\cdot\boldsymbol{y}} \right] |0\rangle$$

There is only one interesting term in this whole thing. The one that has a annihilation creation operator pair.

$$= \int \frac{\mathrm{d}^3 p}{[2\pi]^3} \frac{\mathrm{d}^3 p'}{[2\pi]^3} \frac{1}{2\sqrt{\omega_p \omega_{p'}}} \sum_{\lambda=0}^{3} \sum_{\lambda'=0}^{3} \epsilon_\mu(\boldsymbol{p}, \lambda) \epsilon^*_\nu(\boldsymbol{p}', \lambda') \mathrm{e}^{\mathrm{i}[\boldsymbol{p}\cdot\boldsymbol{x} - \boldsymbol{p}'\cdot\boldsymbol{y}]}$$
$$\left\langle 0 \middle| a_{\boldsymbol{p},\lambda} a^\dagger_{\boldsymbol{p}',\lambda'} \middle| 0 \right\rangle$$

We apply the commutator and discard the occupation number operator term which does not contribute anything in a vacuum matrix element.

$$= -\int \frac{\mathrm{d}^3 p}{[2\pi]^3} \frac{\mathrm{d}^3 p'}{[2\pi]^3} \frac{1}{2\sqrt{\omega_p \omega_{p'}}} \sum_{\lambda=0}^{3} \sum_{\lambda'=0}^{3} \epsilon_\mu(\boldsymbol{p}, \lambda) \epsilon^*_\nu(\boldsymbol{p}', \lambda') \mathrm{e}^{\mathrm{i}[\boldsymbol{p}\cdot\boldsymbol{x} - \boldsymbol{p}'\cdot\boldsymbol{y}]}$$
$$[2\pi]^3 \delta^{(3)}(\boldsymbol{p} - \boldsymbol{p}') \eta_{\lambda\lambda'}$$

We can perform a lot of simplifications here. First we integrate over $\boldsymbol{p}'$ to eliminate it.

$$= -\int \frac{\mathrm{d}^3 p}{[2\pi]^3} \frac{1}{2\omega_p} \sum_{\lambda=0}^{3} \sum_{\lambda'=0}^{3} \eta_{\lambda\lambda'} \epsilon_\mu(\boldsymbol{p}, \lambda) \epsilon^*_\nu(\boldsymbol{p}, \lambda') \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot[\boldsymbol{x} - \boldsymbol{y}]}$$

Next we sum over λ' to eliminate that.

$$= -\int \frac{\mathrm{d}^3 p}{[2\pi]^3} \frac{1}{2\omega_p} \sum_{\lambda=0}^{3} \eta_{\lambda\lambda} \epsilon_\mu(\boldsymbol{p}, \lambda) \epsilon^*_\nu(\boldsymbol{p}, \lambda) \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot[\boldsymbol{x} - \boldsymbol{y}]}$$

Then we can recognize the completeness relations of the polarization vectors.

$$= -\eta_{\mu\nu} \int \frac{\mathrm{d}^3 p}{[2\pi]^3} \frac{1}{2\omega_p} \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot[\boldsymbol{x} - \boldsymbol{y}]}$$

This is the desired expression except a minus sign in the exponential function. This can be inserted since the integration domain over $\boldsymbol{p}$ is symmetric around the origin and ω_p only depends on the magnitude of $\boldsymbol{p}$. The ω_p stops us from performing the $\boldsymbol{p}$ integration and get something in the form of a δ-distribution.

Feynman propagator Peskin and Schroeder (1995, pp. 27 – 31) derive the Feynman propagator from the Heisenberg picture propagator for the Klein-Gordon field. Since the components of the electromagnetic field behave like a massless Klein-Gordon field, we can just set the mass to zero and take the exact same derivation in principle.

This problem works with three and four-vectors, which are not differentiated at all in the problem set and by Peskin and Schroeder (ibid.). This forces one to distinguish them from the context, which is hard when the context is defined by the distinction. We therefore stick with ISO 80000-2 and have the three-vector $\boldsymbol{p}$ (with serifs) and the four-vector $\boldsymbol{\mathsf{p}}$ (without serifs).

For the theory to conserve causality we need that $D_\mathrm{F}^{\mu\nu}(\boldsymbol{\mathsf{x}} - \boldsymbol{\mathsf{y}})$ vanishes for space-like intervals $\boldsymbol{\mathsf{x}} - \boldsymbol{\mathsf{y}}$. The matrix element that we have computed in the first part of this subproblem does not vanish for space-like intervals. This means that photons and anti-photons can propagate off-shell. However, this is not what is of interest to us. Causality means that no information can travel on space-like paths. We have to look at the commutator for causal interaction. The canonical quantization defined this to be a complex number, so we can write it as a matrix element without changing anything.

$$[A^\mu(\boldsymbol{\mathsf{x}}), A^\nu(\boldsymbol{\mathsf{y}})] = \langle 0 | [A^\mu(\boldsymbol{\mathsf{x}}), A^\nu(\boldsymbol{\mathsf{y}})] | 0 \rangle$$

We can expand the commutator.

$$= \langle 0 | A^\mu(\boldsymbol{x}) A^\nu(\boldsymbol{y}) | 0 \rangle - \langle 0 | A^\nu(\boldsymbol{y}) A^\mu(\boldsymbol{x}) | 0 \rangle$$

The two summands have been computed in the part before, we just insert this.

$$= \eta^{\mu\nu} \int \frac{\mathrm{d}^3 p}{[2\pi]^3} \frac{1}{2\omega_p} \left[\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot[x-y]} - \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot[x-y]} \right]$$

The crucial trick is to write this as a four-dimensional integral. We slip in the p^0 by writing ω_p really complicated as the evaluation of integral domains. We can also smuggle another minus sign into the exponential function since the domain of integration is symmetric around the origin and a sign change in front of the spatial components of $\boldsymbol{p}$ does not change anything.

$$= \eta^{\mu\nu} \int \frac{\mathrm{d}^3 p}{[2\pi]^3} \left[\left[\frac{1}{2p^0} \mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot[x-y]} \right]_{p_0=\omega_p} + \left[\frac{1}{2p^0} \mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot[x-y]} \right]_{p_0=-\omega_p} \right]$$

Although nothing changed really, this looks like the evaluation of an integrated function at the bounds of the integral. We want to have an unbounded integral over p^0 such that the integration domain is the whole of $\mathbf{R}^4$. An unbounded integral and two evaluations at finite points suggests the use of the residue theorem. The function will have poles in p^0 at two points. We have $\boldsymbol{p}^2 = [p^0]^2 - \boldsymbol{p}^2$. This is also the mass of the particles, which is zero. So we know that $[p^0]^2 = \boldsymbol{p}^2$. A function with p^2 in the denominator will therefore have poles at $p^0 = \pm|\boldsymbol{p}| = \pm\omega_p$. Instead of moving the contour, we move the poles a tiny bit with $\mathrm{i}\epsilon$ and $\epsilon > 0$ a small number. Our singular function then is

$$\frac{1}{\boldsymbol{p}^2 + \mathrm{i}\epsilon} = \frac{1}{[p^0]^2 - \boldsymbol{p}^2 + \mathrm{i}\epsilon}$$

which has poles at $p^0 = \pm[\omega_p - \mathrm{i}\epsilon]$. (Square this relation, omit terms $\mathrm{O}(\epsilon^2)$ and you get the above expression.) One can then check the residuals of this singular function and obtain that they are $\pm 1/\omega_p$. Depending on the sign of $[x^0 - y^0]$ we need to close the contour above or below the real line. We need the exponential suppression such that the great circles do not contribute anything and the only get the real line integral when adding up the residuals.

$$= \eta^{\mu\nu} \int \frac{\mathrm{d}^3 p}{[2\pi]^3} \frac{p^0}{2\pi\mathrm{i}} \frac{1}{\boldsymbol{p}^2 + \mathrm{i}\epsilon} \mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot[x-y]}$$

Now we can just shuffle the factors around and get a form which is the same as the one on the problem set.

$$= \eta^{\mu\nu} \int \frac{\mathrm{d}^4 p}{[2\pi]^4} \frac{-\mathrm{i}}{\boldsymbol{p}^2 + \mathrm{i}\epsilon} \mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot[x-y]}$$

Since the contour integral splits into two parts, where we can only use one of them, one can write it shorter using the time ordering operator T.

References

Peskin, Michael E. and Daniel V. Schroeder (1995). *An Introduction to Quantum Field Theory*. Westview Press. ISBN: 978-0-201-50397-5.