Disclaimer

physics755 – Quantum Field Theory

Problem Set 5

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problem	achieved points	possible points
The Dirac representation		7
Classical solutions		8
total		15

1 The Dirac representation

1.1 Transformation of Weyl spinors

We had transformations with $\boldsymbol{L}$ and $\boldsymbol{K}$, ones with $\boldsymbol{J}_+$ and $\boldsymbol{J}_-$ and lastly ones with $\boldsymbol{\sigma}$. So we look at those new $\boldsymbol{\sigma}^{\mu\nu}$. First the time component:

$$\sigma^{00} = 0$$

The mixed components:

$$\begin{aligned}\sigma^{0i} &= \frac{\mathrm{i}}{4}\left[\sigma^0\bar{\sigma}^i - \sigma^i\bar{\sigma}^0\right] \\ &= \frac{\mathrm{i}}{4}\left[-\sigma^i - \sigma^i\right] \\ &= -\frac{\mathrm{i}}{2}\sigma^i \\ \bar{\sigma}^{0i} &= -\frac{\mathrm{i}}{2}\bar{\sigma}^i\end{aligned}$$

The spatial components:

$$
\begin{aligned}
\sigma^{ij} &= \frac{\mathrm{i}}{4}\left[\sigma^i\bar{\sigma}^j - \sigma^i\bar{\sigma}^j\right] \\
&= -\frac{\mathrm{i}}{4}\left[\sigma^i\sigma^j + \sigma^i\sigma^j\right] \\
&= -\frac{\mathrm{i}}{4}\left[\sigma^i, \sigma^j\right] \\
&= \epsilon_{ijk}\sigma^k \\
\bar{\sigma}^{ij} &= \sigma^{ij}
\end{aligned}
$$

These seem correct since Peskin and Schroeder (1995, (3.26)f) have the same expressions.

Using those identities, we can rewrite the transformation that we previously had for left handed spinors.

$$
\begin{aligned}
-\mathrm{i}\boldsymbol{\Theta}\cdot\frac{\boldsymbol{\sigma}}{2} - \boldsymbol{\beta}\cdot\frac{\boldsymbol{\sigma}}{2} &= -\mathrm{i}\Theta_i\frac{\sigma^i}{2} - \beta_i\frac{\sigma^i}{2} \\
&= -\frac{\mathrm{i}}{4}\left[\Theta_i\epsilon_{ijk}\sigma^{jk} + \beta_i\sigma^{0i}\right]
\end{aligned}
$$

Now one defined the antisymmetric angle tensor $\boldsymbol{\omega}$ such that $\omega_{ij} = \Theta_i\epsilon_{ijk}$ and $\omega_{0i} = \beta_i$. Then we can write

$$
= -\frac{\mathrm{i}}{4}\omega_{\mu\nu}\sigma^{\mu\nu},
$$

which is the desired result for the left handed spinors. For the left handed ones, there was an additional minus in front of $\boldsymbol{\beta}$, such that we need to use $\bar{\sigma}$ there.

1.2 Dirac spinor transformation

We compute the $\boldsymbol{S}$ explicitly, just as done on the lecture on Friday.

$$
\begin{aligned}
S^{\mu\nu} &= \frac{\mathrm{i}}{4}[\gamma^\mu, \gamma^\nu] \\
&= \frac{\mathrm{i}}{4}[\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu] \\
&= \frac{\mathrm{i}}{4}\begin{pmatrix} \sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu & 0 & \\ 0 & \bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu & 0 \end{pmatrix} \\
&= \operatorname{diag}(\sigma^{\mu\nu}, \bar{\sigma}^{\mu\nu})
\end{aligned}
$$

So we see that this commutator of the Dirac matrices in the chiral representation give a block diagonal matrix. In the exponential map, this will still be a block diagonal matrix. As we have shown in the previous part of this problem, the left and right handed parts of the Dirac spinor transform independently, just like the block diagonal form that we have here.

1.3 Commutator

We want to use anticommutation relations to yield $\boldsymbol{\eta}$s which are needed for the final result. So we start with the given commutator.

$$
\begin{aligned}
[\gamma^\mu, \boldsymbol{S}^{\rho\sigma}] &= \frac{\mathrm{i}}{4}[\gamma^\mu, \gamma^\rho\gamma^\sigma - \gamma^\sigma\gamma^\rho] \\
&= \frac{\mathrm{i}}{4}[\gamma^\mu\gamma^\rho\gamma^\sigma - \gamma^\mu\gamma^\sigma\gamma^\rho - \gamma^\rho\gamma^\sigma\gamma^\mu + \gamma^\sigma\gamma^\rho\gamma^\mu]
\end{aligned}
$$

We now use the anticommutation relation of the Dirac matrices.

$$
= \frac{\mathrm{i}}{4}\left[\left[-\gamma^\rho\gamma^\mu + 2\eta^{\mu\rho}\boldsymbol{1}_4\right]\gamma^\sigma - \left[-\gamma^\sigma\gamma^\mu + 2\eta^{\mu\sigma}\boldsymbol{1}_4\right]\gamma^\rho - \gamma^\rho\gamma^\sigma\gamma^\mu + \gamma^\sigma\gamma^\rho\gamma^\mu\right]
$$

Then we expand the inner brackets.

$$
= \frac{\mathrm{i}}{4}\left[-\gamma^\rho\gamma^\mu\gamma^\sigma + \gamma^\sigma\gamma^\mu\gamma^\rho - \gamma^\rho\gamma^\sigma\gamma^\mu + \gamma^\sigma\gamma^\rho\gamma^\mu + 2\eta^{\mu\rho}\boldsymbol{1}_4\gamma^\sigma - 2\eta^{\mu\sigma}\boldsymbol{1}_4\gamma^\rho\right]
$$

We do the same thing again to move the γ^μ to the back.

$$
\begin{aligned}
= \frac{\mathrm{i}}{4}\Big[&-\gamma^\rho\left[-\gamma^\sigma\gamma^\mu + 2\eta^{\mu\sigma}\boldsymbol{1}_4\right] + \gamma^\sigma\left[-\gamma^\rho\gamma^\mu + 2\eta^{\mu\rho}\boldsymbol{1}_4\right] - \gamma^\rho\gamma^\sigma\gamma^\mu + \gamma^\sigma\gamma^\rho\gamma^\mu \\
&+2\eta^{\mu\rho}\boldsymbol{1}_4\gamma^\sigma - 2\eta^{\mu\sigma}\boldsymbol{1}_4\gamma^\rho\Big]
\end{aligned}
$$

And we factor out again.

$$
\begin{aligned}
= \frac{\mathrm{i}}{4}\Big[&\gamma^\rho\gamma^\sigma\gamma^\mu + \gamma^\sigma\gamma^\rho\gamma^\mu - \gamma^\rho\gamma^\sigma\gamma^\mu + \gamma^\sigma\gamma^\rho\gamma^\mu \\
&+2\eta^{\mu\rho}\boldsymbol{1}_4\gamma^\sigma - 2\eta^{\mu\sigma}\boldsymbol{1}_4\gamma^\rho - \gamma^\rho\eta^{\mu\sigma}\boldsymbol{1}_4 + \gamma^\sigma\eta^{\mu\rho}\boldsymbol{1}_4\Big]
\end{aligned}
$$

The first four terms cancel.

$$
= \frac{\mathrm{i}}{4}\left[2\eta^{\mu\rho}\boldsymbol{1}_4\gamma^\sigma - 2\eta^{\mu\sigma}\boldsymbol{1}_4\gamma^\rho - \gamma^\rho 2\eta^{\mu\sigma}\boldsymbol{1}_4 + \gamma^\sigma 2\eta^{\mu\rho}\boldsymbol{1}_4\right]
$$

The components of the matrix tensor are just numbers and commute with the matrices.

$$
= \frac{\mathrm{i}}{4}[2\eta^{\mu\rho}\gamma^\sigma - 2\eta^{\mu\sigma}\gamma^\rho - 2\eta^{\mu\sigma}\gamma^\rho + 2\eta^{\mu\rho}\gamma^\sigma]
$$

Then we can write this shorter by combining terms.

$$
= \mathrm{i}[\eta^{\mu\rho}\gamma^\sigma - \eta^{\mu\sigma}\gamma^\rho]
$$

We can move the Dirac matrix out of the bracket and introduce a Kronecker symbol for it.

$$
= \mathrm{i}\left[\eta^{\mu\rho}\delta^\sigma_\lambda - \eta^{\mu\sigma}\delta^\rho_\lambda\right]\gamma^\lambda
$$

And that is the expression given on the problem set.

1.4 Lorentz invariance of Dirac's equation

This problem is exactly covered by Peskin and Schroeder (1995, p. 42).

Even though $\partial\!\!\!/ = \gamma^\mu \partial_\mu$ *looks like* it would transform like a Lorentz scalar, we cannot assume that. We need to transform the individual parts and then look at the whole thing. The Dirac equation is given by

$$\left[\mathrm{i}\gamma^\mu \partial_\mu - m\right] \psi(x) = 0.$$

Now we transform it passively by adding transformations.

$$\left[\mathrm{i}\gamma^\mu {\Lambda^{-1}}^\nu{}_\mu \partial_\nu - m\right] \Lambda_\mathrm{D} \psi(\Lambda^{-1} x)$$

We premuliply with $\Lambda_\mathrm{D}\Lambda_\mathrm{D}^{-1}$.

$$= \Lambda_\mathrm{D}\Lambda_\mathrm{D}^{-1} \left[\mathrm{i}\gamma^\mu {\Lambda^{-1}}^\nu{}_\mu \partial_\nu - m\right] \Lambda_\mathrm{D} \psi(\Lambda^{-1} x)$$

Since m is just a number, it commutes with the transformations. We can pull some transformations into the bracket, the $\Lambda_\mathrm{D}^{-1}\Lambda_\mathrm{D}$ will just cancel around the m.

$$= \Lambda_\mathrm{D} \left[\Lambda_\mathrm{D}^{-1} \mathrm{i}\gamma^\mu {\Lambda^{-1}}^\nu{}_\mu \partial_\nu \Lambda_\mathrm{D} - m\right] \psi(\Lambda^{-1} x)$$

The matrix Λ_D depends on the angles and boosts, but not on spacetime. It therefore commutes with the partial derivatives.

$$= \Lambda_\mathrm{D} \left[\Lambda_\mathrm{D}^{-1} \mathrm{i}\gamma^\mu \Lambda_\mathrm{D} {\Lambda^{-1}}^\nu{}_\mu \partial_\nu - m\right] \psi(\Lambda^{-1} x)$$

Now we can use the identity that is given on the problem set.

$$= \Lambda_\mathrm{D} \left[\Lambda^\mu_\rho \mathrm{i}\gamma^\rho {\Lambda^{-1}}^\nu{}_\mu \partial_\nu - m\right] \psi(\Lambda^{-1} x)$$

There are now the normal and inverse transformation, they contract to a Kronecker symbol.

$$= \Lambda_\mathrm{D} \left[\mathrm{i}\delta^\nu_\rho \gamma^\rho \partial_\nu - m\right] \psi(\Lambda^{-1} x)$$

And we are back to a Dirac equation:

$$= \Lambda_\mathrm{D} \left[\mathrm{i}\partial\!\!\!/ - m\right] \psi(\Lambda^{-1} x)$$

Now the transformation acts on the whole Dirac equation, which then is form invariant.

1.5 Klein-Gordon equation

This brings us to Peskin and Schroeder (ibid., p. 43).

The Dirac equation is the square root of the Klein-Gordon equation in the sense of the used Clifford algebra. Hamilton had shown this using quaternions around 1840 (Penrose 2005, p. 619). To get back to the Klein-Gordon equation, one has to *square* the equation in the appropriate sense. The correct way of doing this is to take the modulus squared of the Dirac operator:

$$\left|\mathrm{i}\not\partial - m\right|^2 \psi = \left[-\mathrm{i}\not\partial - m\right]\left[\mathrm{i}\not\partial - m\right]\psi$$

There are only two terms that contribute. One could also use the third binomial formula here.

$$= \left[\not\partial\not\partial + m^2\right]\psi$$

This is in a form that is so compact that one cannot see how to simply this. We undo the "slashing".

$$= \left[\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + m^2\right]\psi$$

Since the partial derivatives of a differentiable function commute (theorem of Schwarz), only the symmetric part of the Dirac matrix tensor product contributes. Since we use the non-idempotent form of the symmetrization and antisymmetrization notation now, we need to explicitly introduce a factor of 1/2.

$$= \left[\frac{1}{2}\gamma^{(\mu}\gamma^{\nu)}\partial_\mu\partial_\nu + m^2\right]\psi$$

Since we take the symmetric part of a tensor times itself, $\boldsymbol{\gamma}\otimes\boldsymbol{\gamma}$, we can also use the anticommutator.

$$= \left[\frac{1}{2}\left[\gamma^\mu, \gamma^\nu\right]_+ \partial_\mu\partial_\nu + m^2\right]\psi$$

The anticommutator is known and we get

$$\begin{aligned} &= \left[\eta^{\mu\nu}\partial_\mu\partial_\nu + m^2\right]\psi \\ &= \left[\Box + m^2\right]\psi. \end{aligned}$$

This is the Klein-Gordon equation.

1.6 Lagrange density

The transformation works like this:

$$\begin{aligned} \bar\phi &= \phi^\dagger\gamma^0 \\ &\mapsto [\Lambda\phi]^\dagger\gamma^0 \\ &= \phi^\dagger\Lambda^\dagger\gamma^0 \end{aligned}$$

The transformation is given by this:

$$= \phi^\dagger \exp\left(\frac{\mathrm{i}}{2}\omega_{\mu\nu}S^{\mu\nu}\right)^\dagger \gamma^0$$
$$= \phi^\dagger \exp\left(-\frac{\mathrm{i}}{2}\omega_{\mu\nu}S^{\mu\nu\dagger}\right)\gamma^0.$$

The matrices S^{ij} commute with γ^0 since these S consist of the product of two Dirac matrices. The constituents of said S anticommute with γ^0 individually, and two anticommutations mean a commutation in total. Since S^{ij} are generators of a compact subgroup in the physicist's convention they are hermitian. The generators S^{0i} are anti-hermitian but anticommute with γ^0. γ^0 commutes with itself, which apparently is not trivial, see `http://physics.stackexchange.com/q/139325/5705`. So the anticommutation together with the antihermitian property gives that it commutes and is hermitian for this particular case. We can therefore commute it and remove the hermitian conjugate.

$$= \phi^\dagger\gamma^0 \exp\left(-\frac{\mathrm{i}}{2}\omega_{\mu\nu}S^{\mu\nu}\right)$$

We can now identify the transformation as an *inverse* transformation.

$$= \bar{\phi}\Lambda_{\mathrm{D}}^{-1}$$

We can compute the equation of motion for ϕ easily using the Euler-Lagrange equation for it. We have $\mathscr{L} = \bar{\psi}[\mathrm{i}\partial\!\!\!/ - m]\psi$. Then the Euler-Lagrange equation is

$$\frac{\partial\mathscr{L}}{\partial\bar{\phi}} - \partial_\mu\frac{\partial\mathscr{L}}{\partial\bar{\phi}_{,\mu}} = 0$$
$$[\mathrm{i}\partial\!\!\!/ - m]\psi = 0.$$

That is the Dirac equation. The derivation can also be performed for ψ, which will give the hermitian conjugate of the Dirac equation. Since we will do that in the next problem anyway, we will not typeset that here.

1.7 Hermitian conjugate

γ^0 is a real, symmetric and therefore hermitian matrix. It commutes with itself and the relation is therefore trivially fulfilled. The γ^i in the chiral representation are antihermitian matrices that anticommute with γ^0. So in total, all the minus signs match and we use that the Dirac matrices are their own inverses.

Using this, we can get the equation of motion for $\bar{\psi}$ from the one for ψ:

$$[\mathrm{i}\partial\!\!\!/ - m]\psi = 0$$
$$\mathrm{i}\gamma^\mu\psi_{,\mu} - m\psi = 0$$

We take the hermitian conjugate.

$$-\mathrm{i}\psi^{\dagger}_{,\mu}\gamma^{\mu\dagger} - m\psi^{\dagger} = 0$$

Now we can use the identity that we have derived earlier.

$$-\mathrm{i}\psi^{\dagger}_{,\mu}\gamma^{0}\gamma^{\mu}\gamma^{0} - m\psi^{\dagger} = 0$$

We identity the first spinor.

$$-\mathrm{i}\bar{\psi}_{,\mu}\gamma^{\mu}\gamma^{0} - m\psi^{\dagger} = 0$$

Now the postmultiply with γ^0 and use that its square is the identity.

$$-\mathrm{i}\bar{\psi}_{,\mu}\gamma^{\mu} - m\bar{\psi} = 0$$

We can now multiply with -1 and bring it into the compact form

$$[\mathrm{i}\partial\!\!\!/ + m]\bar{\phi} = 0.$$

2 Classical solutions

2.1 Constraints

Using the free solutions we have:

$$\begin{aligned}
[\mathrm{i}\partial\!\!\!/ - m]\psi(x) &= 0 \\
[\mathrm{i}\partial\!\!\!/ - m]u(p)\exp(-\mathrm{i}p \cdot x) &= 0 \\
[p\!\!\!/ - m]u(p)\exp(-\mathrm{i}p \cdot x) &= 0 \\
[p\!\!\!/ - m]u(p) &= 0
\end{aligned}$$

We look at $p = (m, 0, 0, 0)$ and insert this. Then we have:

$$\begin{aligned}
m\left[\gamma^0 - \mathbf{1}_4\right]u(p) &= 0 \\
m\begin{pmatrix} -\mathbf{1}_2 & \mathbf{1}_2 \\ \mathbf{1}_2 & -\mathbf{1}_2 \end{pmatrix} u(p) &= 0 \\
m\begin{pmatrix} -\mathbf{1}_2 & \mathbf{1}_2 \\ \mathbf{1}_2 & -\mathbf{1}_2 \end{pmatrix} \begin{pmatrix} \xi \\ \xi \end{pmatrix} &= 0
\end{aligned}$$

And the constraint is fulfilled.

2.2 Boost

We boost with rapidity η. Then we have

$$\Lambda p = \begin{pmatrix} \cosh(\eta)m \\ 0 \\ 0 \\ \sinh(\eta)m \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} E \\ 0 \\ 0 \\ p_3 \end{pmatrix}$$

The norm of those has to be the same, so either way we get $E^2 - p_3^2 = m^2$ which is not a surprise because a classic theory should be on-shell. We also get the following equations:

$$\cosh(\rho)m = E, \quad \sinh(\rho)m = p_3.$$

We can write the trigonometric functions in terms of exponentials.

$$[\exp(\rho) + \exp(-\rho)]\,\frac{m}{2} = E, \quad [\exp(\rho) - \exp(-\rho)]\,m = p_3.$$

Now we can solve this system of equations for the exponentials.

$$\exp(\rho)m = E + p_3, \quad \exp(-\rho)m = E - p_3$$

Taking the positive branch of the square root gives us the following.

$$\exp\left(\frac{\rho}{2}\right)\sqrt{m} = \sqrt{E + p_3}, \quad \exp\left(-\frac{\rho}{2}\right)\sqrt{m} = \sqrt{E - p_3}$$

The transformation of the Dirac spinor ψ can be written as

$$\Lambda_{\mathrm{D}} = \exp\left(-\frac{\eta}{2}\begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}\right)$$

since we already derived the transformations of the left and right handed parts and have shown that they can be combined into a block diagonal form.

We will now go along the lines of Peskin and Schroeder (1995, p. 46). The boosted spinor is then given by:

$$u(p) = \exp\left(-\frac{\eta}{2}\begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}\right)u(p)$$

Since the square of the Pauli matrices are always the identity matrix, this exponential will split up into two terms, one containing the identity and one containing the Pauli matrix. The factors in front will give cosh and sinh. The sinh will have a minus sign since we are passively boosting by $-\eta$.

$$= \left[\cosh\left(\frac{\eta}{2}\right)\begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & \mathbf{1}_2 \end{pmatrix} - \sinh\left(\frac{\eta}{2}\right)\begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}\right]u(p)$$

As the next step one has to write cosh and sinh in terms of exponentials to combine the two summands.

$$= \begin{pmatrix} \exp\left(\frac{\eta}{2}\right)\frac{1-\sigma^3}{2} + \exp\left(-\frac{\eta}{2}\right)\frac{1+\sigma^3}{2} & 0 \\ 0 & \exp\left(\frac{\eta}{2}\right)\frac{1+\sigma^3}{2} + \exp\left(-\frac{\eta}{2}\right)\frac{1-\sigma^3}{2} \end{pmatrix} u(p)$$

We insert the ansatz.

$$= \begin{pmatrix} \exp\left(\frac{\eta}{2}\right)\frac{1-\sigma^3}{2} + \exp\left(-\frac{\eta}{2}\right)\frac{1+\sigma^3}{2} & 0 \\ 0 & \exp\left(\frac{\eta}{2}\right)\frac{1+\sigma^3}{2} + \exp\left(-\frac{\eta}{2}\right)\frac{1-\sigma^3}{2} \end{pmatrix} \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

We can now use the equations that we derived for the exponentials earlier on.

$$= \begin{pmatrix} \sqrt{E+p_3}\frac{1-\sigma^3}{2} + \sqrt{E-p_3}\frac{1+\sigma^3}{2} & 0 \\ 0 & \sqrt{E+p_3}\frac{1+\sigma^3}{2} + \sqrt{E-p_3}\frac{1-\sigma^3}{2} \end{pmatrix} \begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

This matrix multiplication is trivial, so we can just write it as a vector.

$$= \begin{pmatrix} \left[\sqrt{E+p_3}\frac{1-\sigma^3}{2} + \sqrt{E-p_3}\frac{1+\sigma^3}{2}\right]\xi \\ \left[\sqrt{E+p_3}\frac{1+\sigma^3}{2} + \sqrt{E-p_3}\frac{1-\sigma^3}{2}\right]\xi \end{pmatrix}$$

2.3 Simplification

The simplification is also the next step by Peskin and Schroeder (1995, p. 46), they just do not give any intermediate results. So here they are:

$$\sqrt{E+p_3}\frac{\mathbf{1}_2-\sigma^3}{2} + \sqrt{E-p_3}\frac{\mathbf{1}_2+\sigma^3}{2}$$
$$= \sqrt{[E+p_3]\frac{[\mathbf{1}_2-\sigma^3]^2}{4}} + \sqrt{[E-p_3]\frac{[\mathbf{1}_2+\sigma^3]^2}{4}}$$

Computation shows that the fraction is a projection operator.

$$= \sqrt{[E+p_3]\frac{\mathbf{1}_2-\sigma^3}{2}} + \sqrt{[E-p_3]\frac{\mathbf{1}_2+\sigma^3}{2}}$$

In order to make it hard to read, the now do nothing, really.

$$= \sqrt{\left[\sqrt{[E+p_3]\frac{\mathbf{1}_2-\sigma^3}{2}} + \sqrt{[E-p_3]\frac{\mathbf{1}_2+\sigma^3}{2}}\right]^2}$$

We can now apply the first binomial formula.

$$= \sqrt{[E+p_3]\frac{\mathbf{1}_2-\sigma^3}{2} + [E-p_3]\frac{\mathbf{1}_2+\sigma^3}{2} + 2\sqrt{[E+p_3]\frac{\mathbf{1}_2-\sigma^3}{2}}\sqrt{[E-p_3]\frac{\mathbf{1}_2+\sigma^3}{2}}}$$

The last summand does not contribute anything. This can be seen either by seeing that the fractions are two orthogonal projection operators or by actually calculating it using the third binomial theorem, which will then yield $\mathbf{1}_2 - \mathbf{1}_2 = 0$.

$$= \sqrt{[E+p_3]\frac{\mathbf{1}_2-\sigma^3}{2} + [E-p_3]\frac{\mathbf{1}_2+\sigma^3}{2}}$$

Then we can compute all of the eight terms, see that most cancel and obtain just the following.

$$= \sqrt{E\mathbf{1}_2 - p_3\sigma^3}$$

Using $p_0 = E$ we can write this as

$$= \sqrt{p_0\sigma^0 - p_3\sigma^3}.$$

For our p, this can also be written as

$$= \sqrt{p_\mu\sigma^\mu}$$

We write the contraction with indices such that it is clear that the zeroth component is included as well.

The second part has minus signs at the spatial components, so it will come out with $\bar{\sigma}$ instead of σ.

2.4 Lorentz invariance

The first relation.

$$\begin{aligned}[p\cdot\sigma][p\cdot\bar{\sigma}] &= p_\mu\sigma^\mu p_\nu\sigma^\nu \\ &= p_\mu p_\nu\sigma^\mu\sigma^\nu\end{aligned}$$

Since the p are symmetric, only the symmetric part of the tensor product of the Pauli matrices contributes. We can write this with the anticommutator.

$$\begin{aligned}&= \frac{1}{2}p_\mu p_\nu[\sigma^\mu,\sigma^\nu]_+ \\ &= p_\mu p_\nu\eta^{\mu\nu} \\ &= p^2\end{aligned}$$

Then we have:

$$
\begin{aligned}
u^\dagger u &= \xi^\dagger p \cdot \sigma \xi + \xi^\dagger p \cdot \bar{\sigma} \xi \\
&= 2\xi^\dagger E_p \xi \\
&= 2E_p \xi^\dagger \xi
\end{aligned}
$$

Using a parity makes this invariant:

$$
\bar{u} u = 2\xi^\dagger \sqrt{p \cdot \sigma} \sqrt{p \cdot \bar{\sigma}} \xi
$$

And as shown previously, this is just the (rest) mass.

$$
= 2m\xi^\dagger \xi
$$

2.5 Changes for negative frequency

We now have the following constraint:

$$
\begin{aligned}
[\mathrm{i}\not{\partial} - m]\psi(x) &= 0 \\
[\mathrm{i}\not{\partial} - m]u(p)\exp(\mathrm{i}p \cdot x) &= 0 \\
[-\not{p} - m]u(p)\exp(\mathrm{i}p \cdot x) &= 0 \\
[-\not{p} - m]u(p) &= 0
\end{aligned}
$$

So the sign in front of the momentum has changed. So there should be a minus sign in front of the second component, just as if γ^0 had been applied, since the parity connects the left and right handed components.

2.6 Completeness relation

This one is rather easy to show.

$$
u^s \bar{u}_s = \begin{pmatrix} \sqrt{p \cdot \sigma} \xi^s \\ \sqrt{p \cdot \bar{\sigma}} \xi^s \end{pmatrix} \begin{pmatrix} \xi_s^\dagger \sqrt{p \cdot \sigma} \xi^s & \xi_s^\dagger \sqrt{p \cdot \bar{\sigma}} \end{pmatrix}
$$

We perform the tensor product and use the identities that we have previously computed.

$$
= \begin{pmatrix} m\mathbf{1}_2 & p \cdot \sigma \\ p \cdot \bar{\sigma} & m\mathbf{1}_2 \end{pmatrix}
$$

This can be expanded like so:

$$
= \begin{pmatrix} 0 & p_0 \sigma^0 \\ p_0 \bar{\sigma}^0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & p_i \sigma^i \\ p_i \bar{\sigma}^i & 0 \end{pmatrix} + \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & \mathbf{1}_2 \end{pmatrix} m
$$

And then we can put in $\bar{\sigma}$ in terms of σ.

$$= p_0 \begin{pmatrix} 0 & \sigma^0 \\ \sigma^0 & 0 \end{pmatrix} + p_i \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} + \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & \mathbf{1}_2 \end{pmatrix} m$$

Look! The Dirac matrices in the chiral representation.

$$= p_\mu \gamma^\mu + m\mathbf{1}_4$$

2.7 Negative frequency

The negative frequency case must the orthogonal to the positive frequency one. One choice would be (Peskin and Schroeder 1995, (3.62)),

$$v^s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma}\eta^s \\ -\sqrt{p \cdot \bar{\sigma}}\eta^s \end{pmatrix}.$$

References

Penrose, Roger (2005). *Road to Reality*. 1. New York: Alfred A. Knopf. ISBN: 0-679-45443-8.

Peskin, Michael E. and Daniel V. Schroeder (1995). *An Introduction to Quantum Field Theory*. Westview Press. ISBN: 978-0-201-50397-5.