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This problem set is not reviewed by a tutor. This is just what I have turned in.

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[disclaimer]

physics755 – Quantum Field Theory

Problem Set 4

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2015-05-08
Group Tuesday – Ripunjay Acharya

problem	achieved points	possible points
Lorentz algebra 2		15
total		15

1 Lorentz algebra 2

1.1 Rotations and boosts

Rotation The generators were derived on the previous problem set. They are:

$$J_{\rho\sigma} = x_{[\sigma}\partial_{\rho]}$$

where we have used the antisymmetrization notation in the non-idempotent form since that seems to be used in this class. This uses the Mathematician's convention of real antisymmetric generators. To get the Physicists's notation, we add −i to the generators and i into the exponential map.

$$J_{\rho\sigma} = \mathrm{i}x_{[\rho}\partial_{\sigma]}$$

Then L^3 is given by $\mathrm{i}x_{[1}\partial_{2]}$. We found it easier to start with the representation on $\boldsymbol{x}'$ and show that the generator of that passive transformation is the same as the one given here. So we start with the rotation $\boldsymbol{R}(\Theta)$ around the x^3-axis:

$$\Phi(x') = \Phi\left(\boldsymbol{R}^{-1}(\Theta) \cdot \boldsymbol{x}\right)$$

Then we can expand this around $\Theta = 0$.

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\mathrm{d}^n \Phi}{\mathrm{d}\Theta^n}(\boldsymbol{x}) \Theta^n$$

Next we write this in operator form,

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \Theta^n \frac{\mathrm{d}^n}{\mathrm{d}\Theta^n} \Phi(\boldsymbol{x}),$$

and then as an exponential:

$$= \exp\left(\Theta \frac{\mathrm{d}}{\mathrm{d}\Theta}\right) \Phi(\boldsymbol{x}),$$

We see that the (Physicist's) generator is given by

$$T = -\mathrm{i} \frac{\mathrm{d}}{\mathrm{d}\Theta}.$$

Now we need to apply the chain rule for this expression.

$$= -\mathrm{i} \frac{\mathrm{d}x'^{\mu}}{\mathrm{d}\Theta}(0) \frac{\partial}{\partial x'^{\mu}}$$

The $\boldsymbol{x}'$ is given by

$$\boldsymbol{x}' = \boldsymbol{R}^{-1}(\Theta) \cdot \boldsymbol{x}.$$

The explicit form of this inverse passive transformation looks like the active rotation around the x^3-axis:

$$\boldsymbol{R}^{-1}(\Theta) = \begin{pmatrix} 1 & & & \\ & \cos(\Theta) & -\sin(\Theta) & \\ & \sin(\Theta) & \cos(\Theta) & \\ & & & 1 \end{pmatrix}.$$

The derivative with respect to Θ at $\Theta = 0$ is then given by

$$\frac{\partial \boldsymbol{R}^{-1}}{\partial \Theta}(0) = \begin{pmatrix} 0 & & & \\ & 0 & -1 & \\ & 1 & 0 & \\ & & & 0 \end{pmatrix}.$$

Contracting this derivative of $\boldsymbol{R}$ with $\boldsymbol{x}$ to give the derivative of $\boldsymbol{x}'$ gives

$$\frac{\mathrm{d}x'^{\mu}}{\mathrm{d}\Theta}(0) = \begin{pmatrix} 0 \\ -x^2 \\ x^1 \\ 0 \end{pmatrix}.$$

As a last step with put that into the generator expression:

$$T = -\mathrm{i}\frac{\mathrm{d}x'^\mu}{\mathrm{d}\Theta}(0)\frac{\partial}{\partial x'^\mu} = -\mathrm{i}\begin{pmatrix} 0 \\ -x^2 \\ x^1 \\ 0 \end{pmatrix}^\mu \partial_\mu = -\mathrm{i}\left[x^1\partial_2 - x^2\partial_1\right] = \mathrm{i}\left[x_1\partial_2 - x_2\partial_1\right].$$

This is the same generator, so the transformation should be the same.

Boost We have

$$K^3 = J^{03} = \mathrm{i}x_{[0}\partial_{3]} = \mathrm{i}\left[x_0\partial_3 - x_3\partial_0\right]$$

for the generator in the functional representation.

The same derivation as above can be applied to the boost as well. The general inverse passive boost is given by:

$$\boldsymbol{B}(\eta) = \begin{pmatrix} \cosh(\eta) & & & -\sinh(\eta) \\ & 1 & & \\ & & 1 & \\ -\sinh(\eta) & & & \cosh(\eta) \end{pmatrix}.$$

From this, the generator of the matrix is given by:

$$-\mathrm{i}\begin{pmatrix} 0 & & & -1 \\ & 0 & & \\ & & 0 & \\ -1 & & & 0 \end{pmatrix}.$$

Contracting that with x^μ we get:

$$\begin{pmatrix} -x^3 \\ 0 \\ 0 \\ -x^0 \end{pmatrix}.$$

And contracting that with ∂ we get

$$\mathrm{i}[x^0\partial_3 + x^3\partial_0] = \mathrm{i}[x_0\partial_3 - x_3\partial_0].$$

So this checks out.

1.2 Commutators

The definition of the L^i is

$$L^i = \frac{1}{2}\epsilon^{ijk}J_{jk}.$$

This can be written as $L = \star J$ and therefore also as $\star L = J$ which in components is

$$\epsilon_{ijk} L^k = P_{ij}.$$

We are not sure how many indices we may use. We think that it is not possible to show this without indices at all. We will use the commutator $[x^i, \partial_j] = \delta^i_j$ instead of the full commutator of the J that is given on the problem set.

The structure constants of the algebra were given in the lecture on 2015-05-08 as Equation (66):

$$[J_i, J_j] = \mathrm{i}\epsilon_{ijk} J_k$$
$$[K_i, K_j] = -\mathrm{i}\epsilon_{ijk} J_k$$
$$[J_i, K_j] = \mathrm{i}\epsilon_{ijk} K_k$$

And also

$$[J^i_+, J^j_+] = \mathrm{i}\epsilon^{ijk} J^k_+$$
$$[J^i_-, J^j_-] = \mathrm{i}\epsilon^{ijk} J^k_-$$
$$[J^i_+, J^j_-] = 0$$

Commutations among L^i

$$\left[L^i, L^a\right] = \epsilon^{ijk}\epsilon^{abc}\left[x_j \partial_k, x_b \partial_c\right]$$

Now we can use the commutator identity to split this.

$$= \epsilon^{ijk}\epsilon^{abc}\left[x_j\left[\partial_k, x_b\partial_c\right] + \left[x_j, x_b\partial_c\right]\partial_k\right]$$

We need to split this up again, so we flip the signs in order to use the same identity.

$$= -\epsilon^{ijk}\epsilon^{abc}\left[x_j\left[x_b\partial_c, \partial_k\right] + \left[x_b\partial_c, x_j\right]\partial_k\right]$$

We expand again.

$$= -\epsilon^{ijk}\epsilon^{abc}\left[x_j x_b\left[\partial_c, \partial_k\right] + x_j\left[x_b, \partial_k\right]\partial_c + x_b\left[\partial_c, x_j\right]\partial_k + \left[x_b, x_j\right]\partial_k\partial_c\right]$$

We use the commutator of $\boldsymbol{x}$ and $\boldsymbol{\partial}$ mentioned above.

$$= -\epsilon^{ijk}\epsilon^{abc}\left[x_j\delta_{bk}\partial_c - x_b\delta_{cj}\partial_k\right]$$

We absorb the minus sign in the ordering within the bracket.

$$= \epsilon^{ijk}\epsilon^{abc}\left[x_b\delta_{cj}\partial_k - x_j\delta_{bk}\partial_c\right]$$

Then we factor out.

$$= \epsilon^{ijk}\epsilon^{abc}x_b\delta_{cj}\partial_k - \epsilon^{ijk}\epsilon^{abc}x_j\delta_{bk}\partial_c$$

Next we can apply the Kronecker symbol.

$$= \epsilon^{ijk}\epsilon^{ab}{}_j x_b\partial_k - \epsilon^{ijk}\epsilon^{a}{}_k{}^c x_j\partial_c$$

We reorder the indices such that the last indices are contracted.

$$= \epsilon^{kij}\epsilon^{ab}{}_j x_b\partial_k - \epsilon^{ijk}\epsilon^{ca}{}_k x_j\partial_c$$

Then we can contract the two Levi-Civita symbols.

$$= \left[\delta^{ka}\delta^{ib} - \delta^{kb}\delta^{ia}\right]x_b\partial_k - \left[\delta^{ic}\delta^{ja} - \delta^{ia}\delta^{jc}\right]x_j\partial_c$$

Then we get four terms.

$$= \delta^{ka}\delta^{ib}x_b\partial_k - \delta^{kb}\delta^{ia}x_b\partial_k - \delta^{ic}\delta^{ja}x_j\partial_c + \delta^{ia}\delta^{jc}x_j\partial_c$$

And then we can contract those.

$$= x_i\partial_a - \delta^{ia}x_f\partial_f - x_a\partial_i + \delta^{ia}x_f\partial_f$$

The second and third term cancel, the remainder then is just

$$= x^{[i}\partial^{a]}.$$

And we can write this as

$$= -\mathrm{i}\epsilon^{iak}L^k.$$

The structure constant should be just ϵ_{ijk}, not with a minus sign to match the algebra $\mathfrak{su}(2)$.

Commutations among K^i *Missing*

Commutations among J_+^i *Missing*

Commutations among J_-^i *Missing*

Commutation of J_+^i and J_-^i

$$[\boldsymbol{J}_+, \boldsymbol{J}_-]^i = [\boldsymbol{L} + \mathrm{i}\boldsymbol{K}, \boldsymbol{L} - \mathrm{i}\boldsymbol{K}]^i$$

We use the bilinearity of the commutator.

$$= [\boldsymbol{L}, \boldsymbol{L}]^i + [\boldsymbol{L}, -\mathrm{i}\boldsymbol{K}]^i + [\mathrm{i}\boldsymbol{K}, \boldsymbol{L}]^i + [\mathrm{i}\boldsymbol{K}, -\mathrm{i}\boldsymbol{K}]^i$$

Then we extract the factors to the front.

$$= [\boldsymbol{L}, \boldsymbol{L}]^i - \mathrm{i}\,[\boldsymbol{L}, \boldsymbol{K}]^i + \mathrm{i}\,[\boldsymbol{K}, \boldsymbol{L}]^i + [\boldsymbol{K}, \boldsymbol{K}]^i$$

The second and first term are the same, so we can join them together. The first and last commutator contain the exact same element on the left and right side, so they are zero.

$$= -2\mathrm{i}\,[\boldsymbol{L}, \boldsymbol{K}]^i$$

Now we can expand the definitions.

$$= 2\mathrm{i}\left[\frac{1}{2}\epsilon^{ijk}P_{jk}, P^{0i}\right]$$

We also expand the P.

$$= 2\mathrm{i}\left[\frac{1}{2}\epsilon^{ijk}x_{[j}\partial_{k]}, x^{[0}\partial^{i]}\right]$$

The Levi-Civita symbol acting on the antisymmetric tensor in the first argument gives a factor of 2. We expand the second argument

$$= 2\mathrm{i}\epsilon^{ijk}\left[x_j\partial_k, x^0\partial^i - x^i\partial^0\right]$$

Now use the bilinearity again.

$$= 2\mathrm{i}\epsilon^{ijk}\left[\left[x_j\partial_k, x^0\partial^i\right] - \left[x_j\partial_k, \partial^0 x^i\right]\right]$$

Since there is only one time like component in each term, we can pull that out of the commutator.

$$= 2\mathrm{i}\epsilon^{ijk}\left[\left[x_j\partial_k, \partial^i\right]x^0 - \left[x_j\partial_k, x^i\right]\partial^0\right]$$

Then we can also pull out the ∂_k in the first term and the x_j in the second.

$$= 2\mathrm{i}\epsilon^{ijk}\left[\left[x_j, \partial^i\right]\partial_k x^0 - x_j\left[\partial_k, x^i\right]\partial^0\right]$$

The commutators are simple, we now have:

$$= 2\mathrm{i}\epsilon^{ijk}\left[\delta^i_j\partial_k x^0 + x_j\delta^i_k\partial^0\right].$$

We can now replace the indices j and k with i in each term. There is no summation on the is!

$$= 2\mathrm{i}\epsilon^{iik}\delta^i_j x^0 + 2\mathrm{i}\epsilon^{iji}x_j\partial^0$$

Since the Levi-Civita symbol is antisymmetric and the index i appears twice, this is just zero.

$$= 0$$

Therefore $\boldsymbol{J}_+$ and $\boldsymbol{J}_-$ commute.

1.3 Left and right-handed spinors

Dimension Using the given formula, the dimension is

$$\left[2 \cdot \frac{1}{2} + 1\right][2 \cdot 0 + 1] = 2 \cdot 1 = 2.$$

Those matrices act on vectors from $\mathbf{C}^2$, which has 2 (complex) dimensions.

Action on left/right-handed spinor For a simple scalar field we had the Lorentz transformation given in Equation (4) on the problem set:

$$\Phi \mapsto \exp(-\mathrm{i}\boldsymbol{\Theta} \cdot \boldsymbol{L} - \mathrm{i}\boldsymbol{\beta} \cdot \boldsymbol{K})\,\Phi$$

Using the definitions of $\boldsymbol{J}_+$ and $\boldsymbol{J}_-$, we can rewrite $\boldsymbol{L}$ and $\boldsymbol{K}$ in terms of those:

$$\boldsymbol{L} = \boldsymbol{J}_+ + \boldsymbol{J}_-, \quad \boldsymbol{K} = \frac{1}{\mathrm{i}}[\boldsymbol{J}_+ - \boldsymbol{J}_-]$$

This lets us rewrite the mapping.

$$\begin{aligned}\Phi &\mapsto \exp(-\mathrm{i}\boldsymbol{\Theta} \cdot \boldsymbol{L} - \mathrm{i}\boldsymbol{\beta} \cdot \boldsymbol{K})\,\Phi \\ &= \exp\left(-\mathrm{i}\boldsymbol{\Theta} \cdot [\boldsymbol{J}_+ + \boldsymbol{J}_-] - \boldsymbol{\beta} \cdot [\boldsymbol{J}_+ - \boldsymbol{J}_-]\right)\,\Phi\end{aligned}$$

We have shown earlier that the $\boldsymbol{J}_+$ and $\boldsymbol{J}_-$ commute with each other. This allows us to split the exponent into two parts.

$$= \exp\left(-\mathrm{i}\boldsymbol{\Theta} \cdot \boldsymbol{J}_+ - \boldsymbol{\beta} \cdot \boldsymbol{J}_+\right)\exp\left(-\mathrm{i}\boldsymbol{\Theta} \cdot \boldsymbol{J}_- + \boldsymbol{\beta} \cdot \boldsymbol{J}_-\right)\,\Phi$$

There are two parts here, belonging to different representations of the Lorentz group. We therefore have for the left and right-handed spinors:

$$\psi_{\mathrm{L}} \mapsto \exp\left(-\mathrm{i}\boldsymbol{\Theta} \cdot \boldsymbol{J}_- + \boldsymbol{\beta} \cdot \boldsymbol{J}_-\right)\,\psi_{\mathrm{L}}, \quad \psi_{\mathrm{R}} \mapsto \exp\left(-\mathrm{i}\boldsymbol{\Theta} \cdot \boldsymbol{J}_+ - \boldsymbol{\beta} \cdot \boldsymbol{J}_+\right)\,\psi_{\mathrm{R}}$$

1.4 Connection of left and right-handed transformations

First identity Show that $\boldsymbol{\sigma}^2\boldsymbol{\sigma}^i\boldsymbol{\sigma}^2 = -\boldsymbol{\sigma}^{i*}$.

Second identity Show that $\boldsymbol{\sigma}^2 \Lambda_\mathrm{L}^* \boldsymbol{\sigma}^2 = \Lambda_\mathrm{R}$. This Λ_L is the result from the previous subsection:

$$\Lambda_\mathrm{L} = \exp\left(-\mathrm{i}\boldsymbol{\Theta} \cdot \boldsymbol{J}_- + \boldsymbol{\beta} \cdot \boldsymbol{J}_-\right)$$

We now show the identity:

$$\boldsymbol{\sigma}^2 \Lambda_\mathrm{L}^* \boldsymbol{\sigma}^2 = \boldsymbol{\sigma}^2 \exp\left(-\mathrm{i}\boldsymbol{\Theta} \cdot \boldsymbol{J}_- + \boldsymbol{\beta} \cdot \boldsymbol{J}_-\right)^* \boldsymbol{\sigma}^2$$

We insert $\boldsymbol{J}_- = \boldsymbol{\sigma}/2$.

$$= \boldsymbol{\sigma}^2 \exp\left(\mathrm{i}\boldsymbol{\Theta} \cdot \frac{\boldsymbol{\sigma}^*}{2} + \boldsymbol{\beta} \cdot \frac{\boldsymbol{\sigma}^*}{2}\right) \boldsymbol{\sigma}^2$$

We write the exponential as a power series.

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \boldsymbol{\sigma}^2 \left[\mathrm{i}\boldsymbol{\Theta} \cdot \frac{\boldsymbol{\sigma}^*}{2} + \boldsymbol{\beta} \cdot \frac{\boldsymbol{\sigma}^*}{2}\right]^n \boldsymbol{\sigma}^2$$

We factor out the Pauli matrices.

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \boldsymbol{\sigma}^2 \left[\left[\mathrm{i}\boldsymbol{\Theta} + \boldsymbol{\beta}\right] \cdot \frac{\boldsymbol{\sigma}^*}{2}\right]^n \boldsymbol{\sigma}^2$$

Now we make it a bit more complicated and write the power with a product sign. In order to stay out of trouble with $n = 0$, we split this off.

$$= \mathbf{1}_2 + \sum_{n=1}^{\infty} \frac{1}{n!} \boldsymbol{\sigma}^2 \left[\prod_{k=1}^{n} \left[\mathrm{i}\boldsymbol{\Theta} + \boldsymbol{\beta}\right] \cdot \frac{\boldsymbol{\sigma}^*}{2}\right] \boldsymbol{\sigma}^2$$

Since all the Pauli matrices are their own inverses, one can just add $\boldsymbol{\sigma}^2\boldsymbol{\sigma}^2$ to every factor. This means that we can extend the scope of the product. All the intermediate $\boldsymbol{\sigma}^2$ will cancel each other pairwise.

$$= \mathbf{1}_2 + \sum_{n=1}^{\infty} \frac{1}{n!} \prod_{k=1}^{n} \sigma^2 \left[\mathrm{i}\boldsymbol{\Theta} + \boldsymbol{\beta}\right] \cdot \frac{\boldsymbol{\sigma}^*}{2} \boldsymbol{\sigma}^2$$

However, we can now use the first relation on each summand in the scalar product. Therefore we get

$$= \mathbf{1}_2 + \sum_{n=1}^{\infty} \frac{1}{n!} \prod_{k=1}^{n} \left[-\mathrm{i}\boldsymbol{\Theta} - \boldsymbol{\beta}\right] \cdot \frac{\boldsymbol{\sigma}}{2}.$$

One can write the product as a plain power again, add the $n = 0$ case back to the sum and write it as an exponential.

$$= \exp\left([-i\boldsymbol{\Theta} - \boldsymbol{\beta}] \cdot \frac{\boldsymbol{\sigma}}{2}\right)$$

This is

$$= \Lambda_{\mathrm{R}}.$$

Left-handed spinor The transformation of a left-handed spinor is given by

$$\psi_{\mathrm{L}} \mapsto \Lambda_{\mathrm{L}}\psi_{\mathrm{L}}.$$

We complex conjugate the whole mapping.

$$\psi_{\mathrm{L}}^* \mapsto \Lambda_{\mathrm{L}}^*\psi_{\mathrm{L}}^*$$

Now we premultiply with $\mathrm{i}\boldsymbol{\sigma}^2$.

$$\mathrm{i}\boldsymbol{\sigma}^2\psi_{\mathrm{L}}^* \mapsto \mathrm{i}\boldsymbol{\sigma}^2\Lambda_{\mathrm{L}}^*\psi_{\mathrm{L}}^*$$

Since $\boldsymbol{\sigma}^2\boldsymbol{\sigma}^2 = \mathbf{1}_2$, we can add those between the transformation and the spinor.

$$\mathrm{i}\boldsymbol{\sigma}^2\psi_{\mathrm{L}}^* \mapsto \mathrm{i}\boldsymbol{\sigma}^2\Lambda_{\mathrm{L}}^*\boldsymbol{\sigma}^2\boldsymbol{\sigma}^2\psi_{\mathrm{L}}^*$$

Then we use the second identity that we have shown in this subsection. The i commutes with everything, so we move it to the right position.

$$\mathrm{i}\boldsymbol{\sigma}^2\psi_{\mathrm{L}}^* \mapsto \Lambda_{\mathrm{R}}\boldsymbol{\sigma}^2\mathrm{i}\psi_{\mathrm{L}}^*$$

Now one can see the definition of that ψ_{L}^c.

$$\psi_{\mathrm{L}}^c \mapsto \Lambda_{\mathrm{R}}\psi_{\mathrm{L}}^c$$

We do not know what that "c" stands for. If it is derived from some word, it should have been upright in the previous and following parts.

Right-handed spinor This one is very similar to the left-handed one.

$$\psi_{\mathrm{R}} \mapsto \Lambda_{\mathrm{R}}\psi_{\mathrm{R}}$$

We use the second identity in reverse.

$$\psi_{\mathrm{R}} \mapsto \sigma^2\Lambda_{\mathrm{L}}^*\sigma^2\psi_{\mathrm{R}}$$

Then we premultiply with $-\mathrm{i}\sigma^2$.

$$-\mathrm{i}\sigma^2\psi_\mathrm{R} \mapsto -\mathrm{i}\Lambda_\mathrm{L}^*\sigma^2\psi_\mathrm{R}$$

We complex conjugate the mapping.

$$\mathrm{i}\sigma^{2*}\psi_\mathrm{R}^* \mapsto \mathrm{i}\Lambda_\mathrm{L}\sigma^{2*}\psi_\mathrm{R}^*$$

The complex conjugate of σ^2 is just the negative since it is purely imaginary.

$$-\mathrm{i}\sigma^2\psi_\mathrm{R}^* \mapsto -\mathrm{i}\Lambda_\mathrm{L}\sigma^2\psi_\mathrm{R}^*$$

And now we can identify ψ_R^c like we did before.

$$\psi_\mathrm{R}^c \mapsto \Lambda_\mathrm{L}\psi_\mathrm{R}^c$$

1.5 Hermitian matrices and four-vectors

Determinant The determinant is shown quickly:

$$\begin{aligned}\det(\boldsymbol{x}) &= \left[x^0 + x^3\right]\left[x^0 - x^3\right] - \left[x^1 + \mathrm{i}x^2\right]\left[x^1 - \mathrm{i}x^2\right] \\ &= [x^0]^2 - [x^3]^2 - [x^1]^2 - [x^2]^2 \\ &= [x^0]^2 - [x^1]^2 - [x^2]^2 - [x^3]^2 \\ &= \mathrm{diag}(1,-1,-1,-1)_{\mu\nu}x^\mu x^\nu\end{aligned}$$

And assuming that we use the metric with signature −2, this is

$$= \eta_{\mu\nu}x^\mu x^\nu.$$

Symmetric part We have to show that the symmetric part of $\sigma \otimes \bar{\sigma}$ is given by the twice metric tensor:

$$\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu = 2\eta^{\mu\nu}\boldsymbol{1}_2$$

We lower the index ν.

$$\sigma^\mu\bar{\sigma}_\nu + \sigma_\nu\bar{\sigma}^\mu = 2\delta^\mu_\nu\boldsymbol{1}_2$$

The following hurts a bit, but using the metric with signature −2 we can raise the index ν on again and converting the $\bar{\sigma}$ into σ in that process. We just shift the minus sign around. The index positions on the left and right do not match any more.

$$\sigma^\mu\sigma^\nu + \sigma^\nu\sigma^\mu = 2\delta^\mu_\nu\boldsymbol{1}_2$$

This now is the anticommutation relation of the Pauli matrices. Since $\sigma^i \sigma^i = \mathbf{1}_2$, the $\mu = \nu$ parts have to hold. If μ and ν are different, the product is antisymmetric, so nothing is left. This is evident in the product of the Pauli matrices:

$$\sigma_i \sigma_j = \mathrm{i}\epsilon_{ijk}\sigma_k + \delta_{ij}\mathbf{1}_2.$$

Trace

$$\mathrm{tr}\left(\sigma^\mu \bar{\sigma}_\nu\right) = 2\delta^\mu_\nu$$

First we raise the index of the $\bar{\sigma}$ and remove the bar.

$$\mathrm{tr}\left(\sigma^\mu \sigma^\nu\right) = 2\delta^\mu_\nu$$

To compute the product, we have to look at temporal and spatial parts separately. We write this in the following way: Summands with Latin indices apply when μ and ν are not zero. If that is the case, then $\mu \sim m$ and $\nu \sim n$.

$$\mathrm{tr}\left(\mathrm{i}\epsilon_{mnp}\sigma^m + \delta^{mn}\mathbf{1}_2 + \delta^{\mu 0}\delta^{0\nu}\mathbf{1}_2\right) = 2\delta^\mu_\nu$$

The single Pauli matrix has no trace. The unit matrix has trace 2. So after using the linearity of the trace we are left with the following.

$$2\left[\delta^{mn} + \delta^{\mu 0}\delta^{0\nu}\right] = 2\delta^\mu_\nu$$

All possible values of μ and ν are accounted for and this holds.

Basis and components Now we have to show that $\boldsymbol{x} = \bar{\sigma}_\mu x^\mu$. This can be seen by writing it out:

$$\boldsymbol{x} = x^0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + x^1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + x^2 \begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} + x^3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The sum indeed is the form of $\boldsymbol{x}$ given on the sheet.

Component projection Last we have to show $x^\mu = \frac{1}{2}\mathrm{tr}(\sigma^\mu \boldsymbol{x})$ to get the one point for this problem.

For $\mu = 0$, this clearly is the case since $\sigma^\mu = \mathbf{1}_2$. For other μ we can write

$$x^i = \frac{1}{2}\mathrm{tr}\left(\sigma^i \boldsymbol{x}\right)$$

Then we expand $\boldsymbol{x}$ as shown directly before this problem.

$$= \frac{1}{2}\mathrm{tr}\left(\sigma^i \bar{\sigma}_\nu x^\nu\right)$$

We split temporal and spatial components again, just a little less crude here.

$$= \frac{1}{2} \operatorname{tr}\left(\sigma^i \bar{\sigma}_0 x^0 + \sigma^i \bar{\sigma}_n x^n\right)$$

The first summand is easy since $\bar{\sigma}_0$ is the identity. We can raise the index and remove the bar.

$$= \frac{1}{2} \operatorname{tr}\left(\sigma^i x^0 + \sigma^i \sigma^n x^n\right)$$

The single Pauli matrix is traceless. We use the product expression for the second.

$$= \frac{1}{2} \operatorname{tr}\left(\left[\epsilon^i{}_{np} \sigma^p + \delta^{in} \mathbf{1}_2\right] x^n\right)$$

Again, the single Pauli matrix has no trace, so all is left is the identity with the Kronecker symbol. Since that has trace 2, we are left with

$$= x^n.$$

1.6 Linear map

First identity We start with the relation that we have to show.

$$A \bar{\sigma}_\mu A^\dagger = \bar{\sigma}_\nu \Lambda^\nu{}_\mu$$

Then we premultiply with σ^λ.

$$\sigma^\lambda A \bar{\sigma}_\mu A^\dagger = \sigma^\lambda \bar{\sigma}_\nu \Lambda^\nu{}_\mu$$

Now we take the trace of both sides. This turns out matrix equation into a scalar equation and we loose information. This is probably a bad thing, although we get the right result in the end. This means that there is only probable cause, not a rigid proof.

$$\operatorname{tr}\left(\sigma^\lambda A \bar{\sigma}_\mu A^\dagger\right) = \operatorname{tr}\left(\sigma^\lambda \bar{\sigma}_\nu \Lambda^\nu{}_\mu\right)$$

The left hand side is the definition of Λ.

$$\Lambda^\lambda{}_\mu = \operatorname{tr}\left(\sigma^\lambda \bar{\sigma}_\nu \Lambda^\nu{}_\mu\right)$$

We split up the temporal and spatial parts again. We explicitly write out the minus sign in the spatial parts of $\bar{\sigma}$ instead of fiddling with the raising of the indices. Taking the trace is enough of sketchy tricks for one derivation. We will still use $\lambda \sim l$ and $\nu \sim n$ for the spatial parts of those indices, though. The four cases that we have are:

$$\sigma^0 \sigma_0 = \mathbf{1}_2, \quad -\sigma^0 \sigma_n = -\sigma_n, \quad \sigma^l \sigma_0 = \sigma^l, \quad -\sigma^l \sigma_n = -\mathrm{i}\epsilon_{lnk}\sigma^k + \delta^l_n \mathbf{1}_2.$$

Taking the trace only leaves the first or the last term. We therefore have:

$$\Lambda^\lambda{}_\mu = \mathrm{tr}\left(\left[\delta^{\lambda 0}\delta^{\nu 0} + \delta^{ln}\right]\mathbf{1}_2 \Lambda^\nu{}_\mu\right)$$

This then is the desired result,

$$\Lambda^\lambda{}_\mu = \delta^\lambda_\nu \Lambda^\nu{}_\mu,$$

which is a true statement:

$$\Lambda^\lambda{}_\mu = \Lambda^\lambda{}_\mu.$$

Second identity *Missing*

1.7 Transformation of bilinear

Missing

1.8 Reality condition

First we note that

$$\xi_\mathrm{L} := -\mathrm{i}\sigma^2 \xi^*_\mathrm{R} \iff \xi^*_\mathrm{R} = \mathrm{i}\sigma^2 \xi^*_\mathrm{R}.$$

Now we can tend to the equation we have to derive.

$$\left[\xi^\dagger_\mathrm{R} \sigma^\mu \psi_\mathrm{R}\right]^* = \xi^{\dagger *}_\mathrm{R} \sigma^{\mu *} \psi^*_\mathrm{R}$$

Now we replace the right-handed spinors with the charge conjugated left-handed spinors.

$$= \left[\mathrm{i}\sigma^2 \xi_\mathrm{L}\right]^\dagger \sigma^{\mu *} \left[-\mathrm{i}\sigma^2 \psi_\mathrm{L}\right]$$

The hermitian conjugate of i turns that into its negative. The Pauli matrices are hermitian. We also have $\sigma^{2*} = -\sigma^2$.

$$= -\mathrm{i}\xi^\dagger_\mathrm{L} \sigma^2 \sigma^{\mu *} \mathrm{i}\sigma^2 \psi_\mathrm{L}$$

We can also remove all the imaginary units.

$$= \xi_{\mathrm{L}}^{\dagger} \sigma^2 \sigma^{\mu *} \sigma^2 \psi_{\mathrm{L}}$$

In subproblem 4 we have shown that $\sigma^2 \sigma^i \sigma^2 = -\sigma^{i*}$. The case $\mu = 0$ gives a identity matrix that commutes with σ^2. The result therefore is the identity matrix itself. The expression $\sigma^2 \sigma^{\mu *} \sigma^2$ therefore either is σ^0 or $-\sigma^m$. That is nicely wrapped up in $\bar{\sigma}^{\mu}$. We now have the final result:

$$= \xi_{\mathrm{L}}^{\dagger} \bar{\mu}^{\nu} \psi_{\mathrm{L}}.$$