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physics755 – Quantum Field Theory

Problem Set 2

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Group Tuesday – Ripunjay Acharya

problem	achieved points	possible points
Poincaré algebra		10
total		10

1 Poincaré algebra

1.1 Generators of SO(1, 3) and SO(3)

The $m^{\mu}{}_{\nu}$ and $t^{i}{}_{j}$ are the generators in the Physicist's convention of hermitian generators since the Taylor expansion contains an imaginary unit.

Orthogonal transformations from SO(1, 3), which are Lorentz transformations or from SO(3), which are rotations in three dimensions, have to leave the metric tensor η invariant. I will solve this for the general case of SO(p, q) and relate that to both special cases. The metric of interest in those spaces has signature (p, q), that means p positive and q negative eigenvalues. Since we are only concerned with special relativity here, the metric tensor always has the special form

$$\boldsymbol{\eta} = \operatorname{diag}(\underbrace{1, \dots, 1}_{p \text{ times}}, \underbrace{-1, \dots, -1}_{q \text{ times}}).$$

Let $\boldsymbol{A}$ be the transformation matrix. Then the invariance of the metric is written in index notation as

$$\eta_{ab} = \eta_{ij} [A^{-1}]^{i}{}_{a} [A^{-1}]^{j}{}_{b}.$$

The transformation can now be written in terms of a generator expression with the generators $\boldsymbol{T}_k$ of the group.

$$= \eta_{ij} \exp\left(-\mathrm{i}\epsilon^k \boldsymbol{T}_k\right)^i{}_a \exp\left(-\mathrm{i}\epsilon^k \boldsymbol{T}_k\right)^j{}_b$$

In the first order of $\boldsymbol{\epsilon}$, this can be linearized to give a form like the one given in the problem statement. For those orthogonal groups, there are multiple generators. Here it suffices to look at one of those generators since they all have the same properties. Linear combinations are still generators, the generators form a vector space. This vector space is the tangent space of the group manifold around the identity element. The problem statement basically has $\epsilon \boldsymbol{m} = \epsilon^k \boldsymbol{T}_k$. I will now just look at one of the generators $\boldsymbol{T}$ without an index.

$$= \eta_{ij} \left[\delta^i_a - \mathrm{i}\epsilon T^i{}_a\right]\left[\delta^j_b - \mathrm{i}\epsilon T^j{}_b\right] + \mathrm{O}(\epsilon^2)$$

Then I can expand those brackets. I will drop the fourth summand since it is of second order already.

$$= \eta_{ij}\delta^i_a\delta^j_b - \mathrm{i}\epsilon\eta_{ij}\left[T^i{}_a\delta^j_b + \delta^i_a T^j{}_b\right] + \mathrm{O}(\epsilon^2)$$

I will clean up the indices.

$$= \eta_{ab} - \mathrm{i}\epsilon\left[\eta_{ib}T^i{}_a + \eta_{aj}T^j{}_b\right]$$

This equation now leads to a shorter one:

$$\eta_{ib}T^i{}_a + \eta_{aj}T^j{}_b = 0.$$

I can even pull down the index and write this as:

$$T_{ba} + T_{ab} = 0.$$

Using the (anti)symmetrization notation by Penrose (2005) in the idempotent form, this can be written even shorter as:

$$T_{(ab)} = 0 \iff \frac{1}{2!}\left[T_{ab} + T_{ba}\right] = 0.$$

The left side can be read as "the symmetric part of $\boldsymbol{T}$ is zero". That means that $\boldsymbol{T}$ is completely antisymmetric, which is what the other equations have been saying all along.

Generators are generally without a trace, which is automatically fulfilled by an antisymmetric tensor. Together with

$$\det(\exp(\boldsymbol{A})) = \exp(\operatorname{tr}(\boldsymbol{A}))$$

the vanishing trace also means that the determinant is unity, which is required by a special Lorentz transformation.

So the requirement for the tensors $\boldsymbol{m}$ and $\boldsymbol{t}$ are that they are antisymmetric when written with all lower indices. Since they also have to be hermitian, they must be purely imaginary.

Lorentz transformation For SO(1, 3) we have $\eta = \operatorname{diag}(1, -1, -1, -1)$ and a 4×4 matrix $\boldsymbol{m}$. Since it is antisymmetric and purely imaginary, there are 6 degrees of freedom, which corresponds to the three rotations and the three pseudo rotations (boosts).

Rotation In SO(3) we have $p = 3$ and $q = 0$. Therefore the metric tensor is trivial with $\eta = \boldsymbol{1}_3$. The matrices have three degrees of freedom which correspond to say the Euler angles.

1.2 Generators for Lorentz group

Corresponding generators The generators $\boldsymbol{m}_i$, where i could also be a more complex index like $\rho\sigma$, are the generators of the matrix representation of the Lorentz group. Those $\boldsymbol{m}_i$ act on $\boldsymbol{x}$ to produce $\boldsymbol{x}'$. The L_i that are asked for are the generators of the functional representation of the same group. They are connected by the Taylor expansion given in Equation (11) on the problem set.

Looking at the infinitesimal transformation of the $\boldsymbol{x}$ I have already given that in the first part of the problem. I amended the formula with an index i to sum over multiple generators. The transformation then is given by

$$x'^\mu = x^\mu + \mathrm{i}\epsilon^i {m_i}^\mu{}_\nu x^\nu.$$

Here, the $\boldsymbol{m}_i$ act as the generators of the Lorentz transformations. Now I take the derivative with respect to the parameter ϵ^i:

$$\frac{\partial x'^\mu}{\partial \epsilon^i} = \mathrm{i}{m_i}^\mu{}_\nu x^\nu.$$

To get the whole generator I need to contract it with ∂_μ:

$$L_i = \frac{\partial x'^\mu}{\partial \epsilon^i}\partial_\mu = \mathrm{i}{m_i}^\mu{}_\nu x^\nu \partial_\mu.$$

Those are the generators in the representation on functions.

Commutation relation of the m The first step is to work out the commutation relation of the $\boldsymbol{m}_i$. This commutator has two free indices because $\boldsymbol{m}_i$ are tensors of valence (1, 1). I can write out the matrix multiplication from the commutator:

$$\left[\boldsymbol{m}_{\mu\nu}, \boldsymbol{m}_{\rho\sigma}\right]^\alpha{}_\beta = {m_{\mu\nu}}^\alpha{}_\gamma {m_{\rho\sigma}}^\gamma{}_\beta - {m_{\rho\sigma}}^\alpha{}_\gamma {m_{\mu\nu}}^\gamma{}_\beta$$

This still fits one one line, so the solution is to expand the m as well in terms of its definition. For that, I will use my idempotent version of the antisymmetrization bracket. This causes the factor of 4 in front of everything.

$$= 4\mathrm{i}\left[\delta^\alpha_{[\mu}\eta_{\nu]\gamma}\delta^\gamma_{[\rho}\eta_{\sigma]\beta} - \delta^\alpha_{[\rho}\eta_{\sigma]\gamma}\delta^\gamma_{[\mu}\eta_{\nu]\beta}\right]$$

Now I can execute the middle Kronecker symbol.

$$= 4\mathrm{i}\left[\delta^\alpha_{[\mu}\eta_{\nu][\rho}\eta_{\sigma]\beta} - \delta^\alpha_{[\rho}\eta_{\sigma][\mu}\eta_{\nu]\beta}\right]$$

Here comes the point where I have to expand the antisymmetrization brackets again. I start with the second bracket.

$$= 2\mathrm{i}\left[\delta^\alpha_{[\mu}\eta_{\nu]\rho}\eta_{\sigma\beta} - \delta^\alpha_{[\mu}\eta_{\nu]\sigma}\eta_{\rho\beta} - \delta^\alpha_{[\rho}\eta_{\sigma]\mu}\eta_{\nu\beta} + \delta^\alpha_{[\rho}\eta_{\sigma]\nu}\eta_{\mu\beta}\right]$$

And then the first bracket as well.

$$\begin{aligned}= \mathrm{i}\Big[&\delta^\alpha_\mu\eta_{\nu\rho}\eta_{\sigma\beta} - \delta^\alpha_\nu\eta_{\mu\rho}\eta_{\sigma\beta} - \delta^\alpha_\mu\eta_{\nu\sigma}\eta_{\rho\beta} + \delta^\alpha_\nu\eta_{\mu\sigma}\eta_{\rho\beta}\\ &- \delta^\alpha_\rho\eta_{\sigma\mu}\eta_{\nu\beta} + \delta^\alpha_\sigma\eta_{\rho\mu}\eta_{\nu\beta} + \delta^\alpha_\rho\eta_{\sigma\nu}\eta_{\mu\beta} - \delta^\alpha_\sigma\eta_{\rho\nu}\eta_{\mu\beta}\Big]\end{aligned}$$

Now I switch the two $\boldsymbol{\eta}$ in each term.

$$\begin{aligned}= \mathrm{i}\Big[&\delta^\alpha_\mu\eta_{\sigma\beta}\eta_{\nu\rho} - \delta^\alpha_\nu\eta_{\sigma\beta}\eta_{\mu\rho} - \delta^\alpha_\mu\eta_{\rho\beta}\eta_{\nu\sigma} + \delta^\alpha_\nu\eta_{\rho\beta}\eta_{\mu\sigma}\\ &- \delta^\alpha_\rho\eta_{\nu\beta}\eta_{\sigma\mu} + \delta^\alpha_\sigma\eta_{\nu\beta}\eta_{\rho\mu} + \delta^\alpha_\rho\eta_{\mu\beta}\eta_{\sigma\nu} - \delta^\alpha_\sigma\eta_{\mu\beta}\eta_{\rho\nu}\Big]\end{aligned}$$

Now those terms need to be regrouped by the indices in the last $\boldsymbol{\eta}$. The metric tensor is symmetric, so the order of the indices there does not make a difference.

$$\begin{aligned}= \mathrm{i}\Big[&\delta^\alpha_\mu\eta_{\sigma\beta}\eta_{\nu\rho} - \delta^\alpha_\sigma\eta_{\mu\beta}\eta_{\rho\nu} + \delta^\alpha_\sigma\eta_{\nu\beta}\eta_{\rho\mu} - \delta^\alpha_\nu\eta_{\sigma\beta}\eta_{\mu\rho}\\ &+ \delta^\alpha_\nu\eta_{\rho\beta}\eta_{\mu\sigma} - \delta^\alpha_\rho\eta_{\nu\beta}\eta_{\sigma\mu} + \delta^\alpha_\rho\eta_{\mu\beta}\eta_{\sigma\nu} - \delta^\alpha_\mu\eta_{\rho\beta}\eta_{\nu\sigma}\Big]\end{aligned}$$

Now I can use the antisymmetrization bracket again for the first terms.

$$= 2\mathrm{i}\left[\delta^\alpha_{[\mu}\eta_{\sigma]\beta}\eta_{\nu\rho} + \delta^\alpha_{[\sigma}\eta_{\nu]\beta}\eta_{\rho\mu} + \delta^\alpha_{[\nu}\eta_{\rho]\beta}\eta_{\mu\sigma} + \delta^\alpha_{[\rho}\eta_{\mu]\beta}\eta_{\sigma\nu}\right]$$

Next is the recognition of the $\boldsymbol{m}$. The first two factors always have the indices α and β. So I can pull those indices out.

$$= \mathrm{i}\left[\boldsymbol{m}_{[\mu\sigma]}\eta_{\nu\rho} + \boldsymbol{m}_{[\sigma\nu]}\eta_{\rho\mu} + \boldsymbol{m}_{[\nu\rho]}\eta_{\mu\sigma} + \boldsymbol{m}_{[\rho\mu]}\eta_{\sigma\nu}\right]^\alpha{}_\beta$$

I move the metric tensor in front of the generator.

$$= \mathrm{i}\left[\eta_{\nu\rho}\boldsymbol{m}_{[\mu\sigma]} + \eta_{\rho\mu}\boldsymbol{m}_{[\sigma\nu]} + \eta_{\mu\sigma}\boldsymbol{m}_{[\nu\rho]} + \eta_{\sigma\nu}\boldsymbol{m}_{[\rho\mu]}\right]^\alpha{}_\beta$$

In order to match the exact notation on the problem set, I will switch indices. The metric tensor is symmetric and the generator is antisymmetric. I copied the right hand side as a reminder as well.

$$\left[\boldsymbol{m}_{\mu\nu}, \boldsymbol{m}_{\rho\sigma}\right]^\alpha{}_\beta = \mathrm{i}\left[\eta_{\nu\rho}\boldsymbol{m}_{[\mu\sigma]} - \eta_{\mu\rho}\boldsymbol{m}_{[\nu\sigma]} - \eta_{\nu\sigma}\boldsymbol{m}_{[\mu\rho]} + \eta_{\mu\sigma}\boldsymbol{m}_{[\nu\rho]}\right]^\alpha{}_\beta$$

This looks very similar to the desired commutator of the L_i.

Commutation relation of the *L* Now that I have the commutation of the matrix generators, I can compute the commutator of the generators on functions.

$$[L_i, L_j] = \left[\mathrm{i}m_i{}^\mu{}_\nu x^\nu \partial_\mu, \mathrm{i}m_j{}^\alpha{}_\beta x^\beta \partial_\alpha\right]$$

As a first step, I remove the two imaginary units and change the order in the commutator.

$$= \left[m_i{}^\alpha{}_\beta x^\beta \partial_\alpha, m_j{}^\mu{}_\nu x^\nu \partial_\mu\right]$$

Then I can write it out in full.

$$= m_i{}^\alpha{}_\beta x^\beta \partial_\alpha m_j{}^\mu{}_\nu x^\nu \partial_\mu - m_j{}^\mu{}_\nu x^\nu \partial_\mu m_i{}^\alpha{}_\beta x^\beta \partial_\alpha$$

The $\boldsymbol{m}$ can be pulled in front of the $\boldsymbol{x}$ and $\boldsymbol{\partial}$ since those do not depend on each other.

$$= m_i{}^\alpha{}_\beta m_j{}^\mu{}_\nu x^\beta \partial_\alpha x^\nu \partial_\mu - m_j{}^\mu{}_\nu m_i{}^\alpha{}_\beta x^\nu \partial_\mu x^\beta \partial_\alpha$$

Next I can rename the Greek indices since they are all summed over.

$$= m_i{}^\alpha{}_\beta m_j{}^\mu{}_\nu x^\beta \partial_\alpha x^\nu \partial_\mu - m_j{}^\alpha{}_\beta m_i{}^\mu{}_\nu x^\beta \partial_\alpha x^\nu \partial_\mu$$

Then I can factor out the $\boldsymbol{x}$ and $\boldsymbol{\partial}$.

$$= \left[m_i{}^\alpha{}_\beta m_j{}^\mu{}_\nu - m_j{}^\alpha{}_\beta m_i{}^\mu{}_\nu\right] x^\beta \partial_\alpha x^\nu \partial_\mu$$

The partial derivatives act on a function $f(\boldsymbol{x})$. One has to keep that in mind then computing the commutator. So

$$\partial_\alpha x^\nu \partial_\mu f(\boldsymbol{x}) = [\partial_\alpha x^\nu]\partial_\mu f(\boldsymbol{x}) + x^\nu \partial_\alpha \partial_\mu f(\boldsymbol{x}) = \delta^\nu_\alpha \partial_\mu f(\boldsymbol{x}) + x^\nu \partial_\alpha \partial_\mu f(\boldsymbol{x}) = \left[\delta^\nu_\alpha f(\boldsymbol{x}) + x^\nu \partial_\alpha\right] \partial_\mu f(\boldsymbol{x}).$$

I use this for the last three factors now.

$$= \left[m_i{}^\alpha{}_\beta m_j{}^\mu{}_\nu - m_j{}^\alpha{}_\beta m_i{}^\mu{}_\nu\right] x^\beta \left[\delta^\nu_\alpha + x^\nu \partial_\alpha\right] \partial_\mu$$

Then I expand the second bracket and apply the Kronecker symbol.

$$= \left[m_i{}^\nu{}_\beta m_j{}^\mu{}_\nu - m_j{}^\nu{}_\beta m_i{}^\mu{}_\nu\right] x^\beta \partial_\mu + \left[m_i{}^\alpha{}_\beta m_j{}^\mu{}_\nu - m_j{}^\alpha{}_\beta m_i{}^\mu{}_\nu\right] x^\beta x^\nu \partial_\alpha \partial_\mu$$

The second bracket is antisymmetric in i and j but symmetric in (α, μ) and $(\beta\,\nu)$. Therefore, the second summand is just zero. Only the first summand remains.

$$= \left[m_i{}^\nu{}_\beta m_j{}^\mu{}_\nu - m_j{}^\nu{}_\beta m_i{}^\mu{}_\nu\right] x^\beta \partial_\mu$$

I switch both of the $\boldsymbol{m}$ in pairs such that the commutator gets removed.

$$= \left[m_j{}^\mu{}_\nu m_i{}^\nu{}_\beta - m_i{}^\mu{}_\nu m_j{}^\nu{}_\beta\right] x^\beta \partial_\mu$$

This is the commutator that I have derived previously. I just have to expand the indices a bit. I set $i = \mu\nu$ and $j = \rho\sigma$ and rename the other indices such that there is no clash in them.

$$= \left[m_{\rho\sigma}{}^{\alpha}{}_{\gamma} m_{\mu\nu}{}^{\gamma}{}_{\beta} - m_{\mu\nu}{}^{\alpha}{}_{\gamma} m_{\rho\sigma}{}^{\gamma}{}_{\beta}\right] x^{\beta} \partial_{\alpha}$$

Now I can write this as the commutator.

$$= \left[\boldsymbol{m}_{\rho\sigma}, \boldsymbol{m}_{\mu\nu}\right]^{\alpha}{}_{\beta} x^{\beta} \partial_{\alpha}$$

The commutator was computed before, I just insert the result now. I just need to add the minus sign again since the order in the commutator is not the same here.

$$= -\mathrm{i}\left[\eta_{\nu\rho} \boldsymbol{m}_{[\mu\sigma]} - \eta_{\mu\rho} \boldsymbol{m}_{[\nu\sigma]} - \eta_{\nu\sigma} \boldsymbol{m}_{[\mu\rho]} + \eta_{\mu\sigma} \boldsymbol{m}_{[\nu\rho]}\right]^{\alpha}{}_{\beta} x^{\beta} \partial_{\alpha}$$

Now I can take the contraction with the vector and the partial derivative into every single term and obtain the $L_{\mu\nu}$ again.

$$= -\left[\eta_{\nu\rho} L_{[\mu\sigma]} - \eta_{\mu\rho} L_{[\nu\sigma]} - \eta_{\nu\sigma} L_{[\mu\rho]} + \eta_{\mu\sigma} L_{[\nu\rho]}\right]$$

I am now missing an imaginary unit, though.

1.3 Translations

Translation generators Using Equation (11), this is a direct application. The transformation is

$$x'^{\mu} = x^{\mu} + a^{\mu}$$

where $\{a^{\mu}\}$ are the four parameters of this transformation. The differential operators for the transformation are then simply:

$$P_{\lambda} = \mathrm{i} \frac{\partial x'^{\mu}}{\partial a^{\lambda}} \partial_{\mu} = \mathrm{i} \delta^{\mu}_{\lambda} \partial_{\mu} = \mathrm{i} \partial_{\lambda}.$$

Momentum commutators The commutators $[P_{\mu}, P_{\nu}]$ are zero because the partial derivatives of differentiable functions commute.

Lorentz and momentum commutator

$$[L_{\mu\nu}, P_{\lambda}] = \left[-m_{\mu\nu}{}^{\alpha}{}_{\beta} x^{\beta} \partial_{\alpha}, \mathrm{i}\partial_{\lambda}\right]$$

The generators $\boldsymbol{m}_i$ do not depend on $\boldsymbol{x}$ an can therefore be moved out of the commutator together with all constants.

$$= -\mathrm{i} m_{\mu\nu}{}^{\alpha}{}_{\beta} \left[x^{\beta} \partial_{\alpha}, \partial_{\lambda}\right]$$

The partial derivatives commute anyway, so that can be moved out of the commutator as well.

$$= -\mathrm{i} m_{\mu\nu}{}^{\alpha}{}_{\beta} \left[x^{\beta}, \partial_{\lambda}\right] \partial_{\alpha}$$

That commutator is well known by now, it is $\mathrm{i}\delta^{\beta}_{\lambda}$.

$$= m_{\mu\nu}{}^{\alpha}{}_{\beta} \delta^{\beta}_{\lambda} \partial_{\alpha}$$

Then the indices can be contracted easily.

$$= m_{\mu\nu}{}^{\alpha}{}_{\lambda} \partial_{\alpha}$$

Now that looks like

$$\frac{\partial L_{\mu\nu}}{\partial x^{\lambda}}$$

or $-\mathrm{i} m_{\mu\nu}{}^{\alpha}{}_{\lambda} P_{\alpha}$. I am not sure which makes more sense here.

Closing of algebra *(Missing)*

1.4 Commutators with Pauli-Lubanski vector

First one

$$[W^{\lambda}, L^{\mu\nu}] = \frac{1}{2} \epsilon^{\lambda\sigma\alpha\beta} [L_{\alpha\beta} P_{\sigma}, L^{\mu\nu}]$$

I now use the commutator identity $[AB, C] = A[B, C] + [A, C]B$.

$$= \frac{1}{2} \epsilon^{\lambda\sigma\alpha\beta} \left[L_{\alpha\beta} [P_{\sigma}, L^{\mu\nu}] + [L_{\alpha\beta}, L^{\mu\nu}] P_{\sigma} \right]$$

Second one I first expand the Pauli-Lubanski vector by its definition.

$$[W^{\lambda}, P^{\sigma}] = \frac{1}{2} \epsilon^{\lambda\rho\mu\nu} [L_{\mu\nu} P_{\rho}, P_{\sigma}]$$

The components of $\boldsymbol{P}$ commute with each other, as shown before. Therefore I can move that out to the back of the expression.

$$= \frac{1}{2} \epsilon^{\lambda\rho\mu\nu} [L_{\mu\nu}, P_{\sigma}] P_{\rho}$$

Now I can recognize the commutator from the previous problem.

$$= -\mathrm{i} \frac{1}{2} \epsilon^{\lambda\rho\mu\nu} m_{\mu\nu}{}^{\alpha}{}_{\sigma} P_{\rho}$$

Then I expand the ***m***.

$$= -\mathrm{i}\epsilon^{\lambda\rho\mu\nu}\delta^{\alpha}_{[\mu}\eta_{\nu]\sigma}P_\alpha P_\rho$$

The Levi-Civita symbol antisymmetrizes in the indices μ and ν already, so I can drop the antisymmetrization bracket.

$$= -\mathrm{i}\epsilon^{\lambda\rho\mu\nu}\delta^{\alpha}_{\mu}\eta_{\nu\sigma}P_\alpha P_\rho$$

I execute the Kronecker symbol.

$$= -\mathrm{i}\epsilon^{\lambda\rho\mu\nu}\eta_{\nu\sigma}P_\mu P_\rho$$

The expression is antisymmetrized by the Levi-Civita symbol in μ and ρ, but the ***P*** at the end are symmetric in those indices. The complete expression is therefore zero.

$$= 0$$

Third one

$$[W^\lambda, W^\sigma] = [\epsilon^{\lambda\rho\mu\nu}L_{\mu\nu}P_\rho, W^\sigma]$$

I pull out the scalar.

$$= \epsilon^{\lambda\rho\mu\nu}[L_{\mu\nu}P_\rho, W^\sigma]$$

As shown just before this commutator, the ***P*** commutes with the ***W***. This can be moved to the end then.

$$= \epsilon^{\lambda\rho\mu\nu}[L_{\mu\nu}, W^\sigma]P_\rho$$

Now I can apply the first commutation relation.

$$= 2\mathrm{i}\epsilon^{\lambda\rho\mu\nu}W^{[\mu}\eta^{\nu]\sigma}P_\rho$$

The Levi-Civita symbol already antisymmetrizes μ and ν, so I drop the extra bracket.

$$= \mathrm{i}\epsilon^{\lambda\rho\mu\nu}W^{\mu}\eta^{\nu\sigma}P_\rho$$

Now I can contract over ν.

$$= \mathrm{i}\epsilon^{\lambda\rho\mu\sigma}W^{\mu}P_\rho$$

The dummy indices have to be renamed, but this is the same expression as given on the problem set.

1.5 Casimir operators

In order to show that P^2 and W^2 are Casimir operators, I show that the commutator with every operator is zero.

P^2 with P Since the components of P commute with each other, this is trivially the case:

$$[P^\mu P_\mu, P_\alpha] = 0.$$

P^2 with W The components of P commute with the components of W, so this equally trivially is

$$[P^\mu P_\mu, W^\alpha] = 0.$$

P^2 with L

$$[P^\mu P_\mu, L_{\lambda\rho}] = P^\mu [P_\mu, L_{\lambda\rho}] + [P^\mu, L_{\lambda\rho}] P_\mu$$

I move indices up and down within the contraction.

$$= P^\mu [P_\mu, L_{\lambda\rho}] + [P_\mu, L_{\lambda\rho}] P^\mu$$

Now I expand the L. The m commute with the P.

$$= \mathrm{i} m_{\lambda\rho}{}^\alpha{}_\mu [P^\mu P_\alpha + P_\alpha P^\mu]$$

The P commute with each other.

$$= 2\mathrm{i} m_{\lambda\rho}{}^\alpha{}_\mu P^\mu P_\alpha$$

Then I expand the m.

$$= 4\mathrm{i} \delta^\alpha_{[\lambda} \eta_{\rho]\mu} P^\mu P_\alpha$$

Now I execute the Kronecker symbol. I also use the metric tensor to lower the index on the first P.

$$= 4\mathrm{i} P_{[\lambda} P_{\rho]}$$

It can be seen clearly that the antisymmetric part of this symmetric tensor product $\boldsymbol{P} \otimes \boldsymbol{P}$.

$$= 0$$

W^2 with P The components of W of P commute with each other, so we have

$$[W^\mu W_\mu, P_\lambda] = 0.$$

W^2 with W

$$[W^\mu W_\mu, W^\lambda] = W_\mu [W^\mu, W^\lambda] + [W^\mu, W^\lambda] W_\mu$$

Then I expand the W^μ inside the commutator.

$$= \mathrm{i}\epsilon^{\mu\lambda\alpha\beta}[W_\mu W_\alpha P_\beta + W_\alpha P_\beta W_\mu]$$

This is symmetric in α and β since the W and P commute. It is antisymmetric in those indices because of the Levi-Civita symbol. Therefore, the whole expression is zero.

$$= 0$$

W^2 with L

$$[W^\mu W_\mu, L_{\lambda\sigma}] = W^\mu [W_\mu, L_{\lambda\sigma}] + [W_\mu, L_{\lambda\sigma}] W^\mu$$

Now I can insert the already computed commutator of a single W with L.

$$= 2\mathrm{i}\left[W^\mu W^{[\sigma}\eta^{\lambda]\mu} + W^{[\sigma}\eta^{\lambda]\mu} W^\mu\right]$$

Then I contract over μ.

$$= 2\mathrm{i}\left[W^{[\lambda} W^{\sigma]} + W^{[\sigma} W^{\lambda]}\right]$$

I switch the indices in the second summand.

$$\begin{aligned} &= 2\mathrm{i}\left[W^{[\lambda} W^{\sigma]} - W^{[\lambda} W^{\sigma]}\right] \\ &= 0 \end{aligned}$$

References

Penrose, Roger (2005). *Road to Reality*. 1. New York: Alfred A. Knopf. ISBN: 0-679-45443-8.