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[disclaimer]

**physics755 – Quantum Field Theory**

# Problem Set 1

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| problem | achieved points | possible points |
|---|---|---|
| The complex scalar field | | 10 |
| total | | 10 |

# 1 The complex scalar field

## 1.1 Hamiltonian

**Conjugate momenta** I have given the following action:

$$S = \int \mathrm{d}^4x \left[\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi\right]. \tag{1}$$

The scoping of the $\partial$ is not perfectly clear, I assume that it will only act on the symbol directly after it. So that the first term in the action density is the four-gradient squared. I will use GR notation for that:

$$S = \int \mathrm{d}^4x \left[\phi^*_{,\mu} \phi^{,\mu} - m^2 \phi^* \phi\right]. \tag{2}$$

Now I use

$$S = \int \mathrm{d}t\, L, \quad L = \int \mathrm{d}^3x\, \mathscr{L} \tag{3}$$

where $\mathscr{L}$ is the Lagrange density and obtain the Lagrange function $L$:

$$L = \int \mathrm{d}^3x \left[\phi^*_{,\mu} \phi^{,\mu} - m^2 \phi^* \phi\right]. \tag{4}$$

The conjugate momentum densities are then:

$$\pi = \frac{\partial L}{\partial \dot\phi} = \dot\phi^*, \quad \pi^* = \frac{\partial L}{\partial \dot\phi^*} = \dot\phi. \tag{5}$$

**Commutation relations** The commutation relations are chosen in a certain way to quantize the theory. I do not think there is anything one can do to derive them right here. The relations are, with $a$ and $b$ labelling multiple fields (Schwabl 2008, (12.3.1)):

$$[\phi_a(x,t), \pi_b(y,t)] = \mathrm{i}\delta_{ab}\delta(x-y)$$
$$[\phi_a(x,t), \phi_b(y,t)] = 0$$
$$[\pi_a(x,t), \pi_b(y,t)] = 0$$

Now one can insert the momenta and yield

$$\left[\phi(x,t), \dot\phi^*(y,t)\right] = \mathrm{i}\delta_{ab}\delta(x-y)$$
$$\left[\phi^*(x,t), \dot\phi(y,t)\right] = \mathrm{i}\delta_{ab}\delta(x-y)$$

by also setting $\phi_a := \phi$ and $\phi_b := \phi^*$ (ibid., (13.2.4)).

**Hamiltonian** The Hamiltonian can then be obtained using a Legendre transformation. I have not an explicit recipe for the complex valued case since Peskin and Schroeder (1995, Chapter 2) only cover the real Klein-Gordon field and Schwabl (2008, Chapter 13.2) does not give the Hamiltonian of the complex Klein-Gordon field. I figured out a way that gives the correct result. Please tell me whether it is the correct reasoning.

The Legendre transformation in classical mechanics is

$$H = \pi_i \dot{q}^i - L. \tag{6}$$

The $i$ denote the finite number of degrees of freedom. In a field theory, the degrees of freedoms are infinite, and therefore one has $q_i \to q(x)$ and $x$ labels the degrees of freedom. Now $q(x)$ is the *one* field in consideration. There is a momentum $\pi(x)$ which also has the same degrees of freedom. There still has to be a sum over all the degrees of freedom, which is now done by integration over $x$.

$$H = \int \mathrm{d}x\, \pi(x)\dot{q}(x) - L. \tag{7}$$

Here we have two fields, $\phi$ and $\phi^*$. Therefore, there are multiple terms in the Hamiltonian. The equation looks really similar:

$$H = \int \mathrm{d}x\, \pi_a(x)\dot{\phi}^a(x) - L. \tag{8}$$

Here, the index $a$ numbers the different fields ($a = 1, 2$). So the index summation is over the different fields. The summing over $i$ is now an integral over $x$.

With this, I can find the Hamiltonian:

$$H = \int \mathrm{d}^3x \left[\pi\dot{\phi} + \pi^*\dot{\phi}^* - \mathscr{L}(\phi, \mathrm{d}\phi(\phi, \pi))\right] \tag{9}$$

Before I insert $\mathscr{L}$, I will replace the time derivative of the fields with the canonical momentum density. This will also make the first two summands equal, since the momenta commute.

$$= \int \mathrm{d}^3x \left[2\pi^*\pi - \mathscr{L}\right] \tag{10}$$

Now I expand $\mathscr{L}$.

$$= \int \mathrm{d}^3x \left[2\pi^*\pi - \phi^*_{,\mu}\phi^{,\mu} + m^2\phi^*\phi\right] \tag{11}$$

Using the metric tensor $\eta^{ii} = -1$ one can separate time and spatial components:

$$= \int \mathrm{d}^3x \left[2\pi^*\pi - \left[\dot{\phi}^*\dot{\phi} - \nabla\phi^* \cdot \nabla\phi\right] + m^2\phi^*\phi\right] \tag{12}$$

The first term in the inner bracket is just $\pi^*\pi$ and cancels out part of the first summand. I am left with

$$= \int \mathrm{d}^3x \left[ \pi^*\pi + \nabla\phi^* \cdot \nabla\phi + m^2\phi^*\phi \right]. \tag{13}$$

This is exactly the Hamiltonian given in Equation (7) on the problem set.

**Heisenberg equations of motion** Now that the Hamiltonian is derived, one can use the time evolution in the Heisenberg picture to get the equations of motion for the two fields. This is plugging in into $\mathrm{i}\dot{O} = [H, O]$ for given operator $O$ (Peskin and Schroeder 1995, (2.44)). I can insert $O = \phi$ and $O = \pi$ to get the Hamiltonian equations of motions. I will start with the $\phi(\boldsymbol{x}, t)$.

$$\mathrm{i}\dot{\phi}(\boldsymbol{x}, t) = [\phi(\boldsymbol{x}, t), H] \tag{14}$$

I insert the Hamiltonian. The integration has to be done over a different variable, I use $x'$. All the functions in the Hamiltonian have arguments $(\boldsymbol{x}', t)$ which I omitted to fit this onto a single line.

$$= \left[ \phi(\boldsymbol{x}, t), \int \mathrm{d}^3x' \left[ \pi^*\pi + \nabla\phi^* \cdot \nabla\phi + m^2\phi^*\phi \right] \right] \tag{15}$$

The integration of $\boldsymbol{x}'$ does not involve $\boldsymbol{x}$ at all, so I can use the linearity of the integral and move it out of the commutator. I now have three individual ones I can work with independently.

$$= \int \mathrm{d}^3x' \left[ [\phi(\boldsymbol{x}, t), \pi^*\pi] + [\phi(\boldsymbol{x}, t), \nabla\phi^* \cdot \nabla\phi] + m^2 [\phi(\boldsymbol{x}, t), \phi^*\phi] \right] \tag{16}$$

(1) In the first term, $\phi$ commutes with $\pi^*$, so I can move this out of the commutator. (2) The $\phi$ commute with each other freely, so this commutator is just zero. (3) $\phi$ commutes with itself at any time, so this is also zero. What remains is

$$= \int \mathrm{d}^3x' \, \pi^* \left[ \phi(\boldsymbol{x}, t), \pi(\boldsymbol{x}', t) \right]. \tag{17}$$

Using the commutation relations that I copied earlier, this can be written as

$$= \int \mathrm{d}^3x' \, \pi^*(\boldsymbol{x}', t) \, \mathrm{i} \, \delta(\boldsymbol{x} - \boldsymbol{x}') \tag{18}$$

Now I perform the integration and obtain

$$= \mathrm{i}\pi^*(\boldsymbol{x}, t). \tag{19}$$

This looks similar to the treatment of the real Klein-Gordon field by Peskin and Schroeder (ibid., p. 25). The time evolution of the complex conjugate $\phi^*$ is given by $\pi(\boldsymbol{x}, t)$ as this whole derivation can be complex conjugated ($H^* = H$).

Now I need to perform an analogue calculation for $\dot{\pi}$:

$$\mathrm{i}\dot{\pi}(\boldsymbol{x},t) = [\pi(\boldsymbol{x},t), ] \tag{20}$$

$$= \left[\pi(\boldsymbol{x},t), \int \mathrm{d}^3x' \left[\pi^*\pi + \boldsymbol{\nabla}\phi^* \cdot \boldsymbol{\nabla}\phi + m^2\phi^*\phi\right]\right] \tag{21}$$

Again I move the integration out of the commutator and yield three independent ones:

$$= \int \mathrm{d}^3x' \left[[\pi(\boldsymbol{x},t), \pi^*\pi] + [\pi(\boldsymbol{x},t), \boldsymbol{\nabla}\phi^* \cdot \boldsymbol{\nabla}\phi] + m^2[\pi(\boldsymbol{x},t), \phi^*\phi]\right] \tag{22}$$

The commutators are now different from the previous case with $\phi$. (1) This commutator is zero, since the momentum density commutes with itself at any point in time. (2) This will need partial integration to isolate $\phi$ which does not commute with $\pi$. (3) The last term is like the first term of the previous derivation.

$$= \boldsymbol{\nabla}\phi^*[\pi(\boldsymbol{x},t), \phi]|_{\boldsymbol{x}'=\partial\mathbf{R}^3} + \int \mathrm{d}^3x' \left[-\triangle\phi^*[\pi(\boldsymbol{x},t), \phi] + m^2\phi^*\left[\pi(\boldsymbol{x},t), \phi(\boldsymbol{x}',t)\right]\right] \tag{23}$$

(1) The surface term in front will vanish on the boundary at infinity because everything is assumed to be normalizable and must therefore decay faster than $1/\sqrt{r}$. I will drop that term. (2) The commutator is the negative of the commutation relation given before. (3) That term is now the same as the second term and I will factor that out.

$$= \int \mathrm{d}^3x' \left[\triangle\phi^* - m^2\phi^*\right][\phi(\boldsymbol{x},t), \pi(\boldsymbol{x},t)] \tag{24}$$

That last remaining commutator gives a δ-distribution again. I integrate over it to remove the $\boldsymbol{x}'$ and I obtain

$$= [\triangle - m^2]\phi^*(\boldsymbol{x},t). \tag{25}$$

This also looks similar to the results of Peskin and Schroeder (1995, p. 25).

**Klein-Gordon equation** Both of those results have to be combined to give the equation of motion for the field $\phi$. Taking the time derivative of the first equation and inserting the second yields

$$\ddot{\phi}(\boldsymbol{x},t) = [\triangle - m^2]\phi(\boldsymbol{x},t). \tag{26}$$

as well as the complex conjugate of this equation. This indeed is the classical Klein-Gordon equation (ibid., (2.7)) and the quantum mechanical one (ibid., (2.45)).

### 1.2 Fourier modes

**Expand into Fourier modes** I will go along the lines of Peskin and Schroeder (1995, § 2.3) here. The expansion of $\phi$ and $\pi$ into Fourier modes is formally given by:

$$\phi(\boldsymbol{x}, t) = \int \frac{\mathrm{d}^3 p}{[2\pi]^3} \exp(\mathrm{i}\boldsymbol{p} \cdot \boldsymbol{x}) \phi(\boldsymbol{p}, t). \tag{27}$$

This is just the back-transform from the momentum space into spatial space (is that even a word?). The real task is to find a representation for $\phi(\boldsymbol{p}, t)$. In momentum space, the Klein-Gordon equation looks just like the equation of the harmonic oscillator (ibid., (2.21)). The field and the momentum can then be written with ladder operators in analogy and the harmonic oscillator (ibid., (2.23)):

$$\phi = \frac{1}{2\omega}[a + a^\dagger], \quad p = -\mathrm{i}\sqrt{\frac{\omega}{2}}[a - a^\dagger], \quad \omega := \sqrt{|\boldsymbol{p}|^2 + m^2}. \tag{28}$$

**Side question**

Why can I do this? Is it because the Hamiltonian looks like the one from the harmonic oscillator and one now thinks about the Klein-Gordan field as a field made up of an infinite number of harmonic oscillators? Is that why Peskin and Schroeder (ibid., (2.25)) then use $a_{\boldsymbol{p}}$ next, because that creates another harmonic oscillator with momentum $\boldsymbol{p}$?

I repeat the expansion of the field into Fourier modes:

$$\phi(\boldsymbol{x}, t) = \int \frac{\mathrm{d}^3 p}{[2\pi]^3} \exp(\mathrm{i}\boldsymbol{p} \cdot \boldsymbol{x}) \phi(\boldsymbol{p}, t). \tag{29}$$

Now I insert the ladder operators. I have not understood why the last exponential has a minus sign in it. It looks like the second summand is the Hermitean conjugate of the first one. This makes $\phi$ self-adjoint and therefore real. Peskin and Schroeder (1995, § 2.3) cover the real Klein-Gordon field, so I am not sure whether this is correct here as well. According to Schwabl (2008, § 13.2) this is not correct here and I have to use independent operators for creation and annihilation. The complex conjugate field will then have the order reversed. I will use the notation and call the second set of ladder operators $b$. Peskin and Schroeder (1995) in contrast to Schwabl (2008) have the minus sign at different positions as well. I will stick with the latter one since that really is for the relativistic Klein-Gordan field.

$$\phi(\boldsymbol{x}, t) = \int \frac{\mathrm{d}^3 p}{[2\pi]^3} \frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}} \left[ a_{\boldsymbol{p}} \exp(-\mathrm{i}\boldsymbol{p} \cdot \boldsymbol{x}) + b_{\boldsymbol{p}}^\dagger \exp(\mathrm{i}\boldsymbol{p} \cdot \boldsymbol{x}) \right] \tag{30}$$

Since the integral is symmetric in $\boldsymbol{p}$, one can factor out the exponential by using $-\boldsymbol{p}$ in those parts.

$$\phi(\boldsymbol{x}, t) = \int \frac{\mathrm{d}^3 p}{[2\pi]^3} \frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}} \left[ a_{-\boldsymbol{p}} + b_{\boldsymbol{p}}^\dagger \right] \exp(\mathrm{i}\boldsymbol{p} \cdot \boldsymbol{x}) \tag{31}$$

The complex conjugate, or rather the Hermitian conjugate of this expression can now be computed to give the other part.

$$\phi^*(\boldsymbol{x}, t) = \int \frac{\mathrm{d}^3 p}{[2\pi]^3} \frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}} \left[ b_{-\boldsymbol{p}} + a_{\boldsymbol{p}}^\dagger \right] \exp(\mathrm{i}\boldsymbol{p} \cdot \boldsymbol{x}) \tag{32}$$

The same thing can then be done for the momentum densities as well:

$$\pi(\boldsymbol{x}, t) = -\mathrm{i} \int \frac{\mathrm{d}^3 p}{[2\pi]^3} \sqrt{\frac{\omega_{\boldsymbol{p}}}{2}} \left[ a_{-\boldsymbol{p}} - b_{\boldsymbol{p}}^\dagger \right] \exp(\mathrm{i}\boldsymbol{p} \cdot \boldsymbol{x}) \tag{33}$$

$$\pi^*(\boldsymbol{x}, t) = -\mathrm{i} \int \frac{\mathrm{d}^3 p}{[2\pi]^3} \sqrt{\frac{\omega_{\boldsymbol{p}}}{2}} \left[ b_{-\boldsymbol{p}} - a_{\boldsymbol{p}}^\dagger \right] \exp(\mathrm{i}\boldsymbol{p} \cdot \boldsymbol{x}) \tag{34}$$

The real number constant $\mathrm{\pi}$ and the momentum density $\pi$ can be told apart since the former is in upright font whereas the latter is in italic font. This is done according to ISO standard 80000-2 (ISO 2009).

**Hamilton operator in ladder operators** The next step is to take those four expressions derived here and build up the Hamilton operator with them. The last version that I had for the Hamiltonian was the following:

$$H = \int \mathrm{d}^3 x \left[ \pi^* \pi + \boldsymbol{\nabla} \phi^* \cdot \boldsymbol{\nabla} \phi + m^2 \phi^* \phi \right]. \tag{35}$$

Now I insert all the four expressions into there. It is crucial to choose different integration variables in the products since those are independent of each other. In the following I just replaced everything with the Fourier expansions, nothing is reordered yet.

$$\begin{aligned}
= \int \mathrm{d}^3 x \Bigg[ \mathrm{i} \int \frac{\mathrm{d}^3 p}{[2\pi]^3} \sqrt{\frac{\omega_{\boldsymbol{p}}}{2}} \left[ b_{-\boldsymbol{p}} - a_{\boldsymbol{p}}^\dagger \right] \exp(\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}) \mathrm{i} \int \frac{\mathrm{d}^3 p'}{[2\pi]^3} \sqrt{\frac{\omega_{\boldsymbol{p}'}}{2}} \left[ a_{-\boldsymbol{p}'} - b_{\boldsymbol{p}'}^\dagger \right] \exp(\mathrm{i}\boldsymbol{p}'\cdot\boldsymbol{x}) \\
+ \boldsymbol{\nabla} \int \frac{\mathrm{d}^3 p}{[2\pi]^3} \frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}} \left[ b_{-\boldsymbol{p}} + a_{\boldsymbol{p}}^\dagger \right] \exp(\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}) \cdot \boldsymbol{\nabla} \int \frac{\mathrm{d}^3 p'}{[2\pi]^3} \frac{1}{\sqrt{2\omega_{\boldsymbol{p}'}}} \left[ a_{-\boldsymbol{p}'} + b_{\boldsymbol{p}'}^\dagger \right] \exp(\mathrm{i}\boldsymbol{p}'\cdot\boldsymbol{x}) \\
+ m^2 \int \frac{\mathrm{d}^3 p}{[2\pi]^3} \frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}} \left[ b_{-\boldsymbol{p}} + a_{\boldsymbol{p}}^\dagger \right] \exp(\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}) \int \frac{\mathrm{d}^3 p'}{[2\pi]^3} \frac{1}{\sqrt{2\omega_{\boldsymbol{p}'}}} \left[ a_{-\boldsymbol{p}'} + b_{\boldsymbol{p}'}^\dagger \right] \exp(\mathrm{i}\boldsymbol{p}'\cdot\boldsymbol{x}) \Bigg]
\end{aligned} \tag{36}$$

Now I can bring terms together and make the whole thing more compact.

$$\begin{aligned}
= \int \mathrm{d}^3 x \Bigg[ - \int \frac{\mathrm{d}^3 p \, \mathrm{d}^3 p'}{[2\pi]^6} \frac{\sqrt{\omega_{\boldsymbol{p}} \omega_{\boldsymbol{p}'}}}{2} \exp\left(\mathrm{i}[\boldsymbol{p}+\boldsymbol{p}']\cdot\boldsymbol{x}\right) \left[ b_{-\boldsymbol{p}} - a_{\boldsymbol{p}}^\dagger \right] \left[ a_{-\boldsymbol{p}'} - b_{\boldsymbol{p}'}^\dagger \right] \\
+ \int \frac{\mathrm{d}^3 p \, \mathrm{d}^3 p'}{[2\pi]^6} \frac{1}{2\sqrt{\omega_{\boldsymbol{p}} \omega_{\boldsymbol{p}'}}} \left[ b_{-\boldsymbol{p}} + a_{\boldsymbol{p}}^\dagger \right] \left[ a_{-\boldsymbol{p}'} + b_{\boldsymbol{p}'}^\dagger \right] \boldsymbol{\nabla} \exp(\mathrm{i}\boldsymbol{p}\boldsymbol{x}) \cdot \boldsymbol{\nabla} \exp(\mathrm{i}\boldsymbol{p}'\boldsymbol{x}) \\
+ m^2 \int \frac{\mathrm{d}^3 p \, \mathrm{d}^3 p'}{[2\pi]^6} \frac{1}{2\sqrt{\omega_{\boldsymbol{p}} \omega_{\boldsymbol{p}'}}} \left[ b_{-\boldsymbol{p}} + a_{\boldsymbol{p}}^\dagger \right] \left[ a_{-\boldsymbol{p}'} + b_{\boldsymbol{p}'}^\dagger \right] \exp\left(\mathrm{i}[\boldsymbol{p}+\boldsymbol{p}']\cdot\boldsymbol{x}\right) \Bigg]
\end{aligned} \tag{37}$$

I can now let the $\boldsymbol{\nabla}$ act on the exponential to give me a $\boldsymbol{p}$ and $\boldsymbol{p}'$ respectively. The integration over the momentum is the same in each summand, so I can pull this out front as well. After the differentiation, the exponentials are the same in every term as well, so I gather those in the first line as well.

$$\begin{aligned}
= \int \mathrm{d}^3 x \, \frac{\mathrm{d}^3 p \, \mathrm{d}^3 p'}{[2\pi]^6} \exp\left(\mathrm{i}[\boldsymbol{p}+\boldsymbol{p}']\cdot\boldsymbol{x}\right) \Bigg[ - \frac{\sqrt{\omega_{\boldsymbol{p}} \omega_{\boldsymbol{p}'}}}{2} \left[ b_{-\boldsymbol{p}} - a_{\boldsymbol{p}}^\dagger \right] \left[ a_{-\boldsymbol{p}'} - b_{\boldsymbol{p}'}^\dagger \right] \\
+ \frac{1}{2\sqrt{\omega_{\boldsymbol{p}} \omega_{\boldsymbol{p}'}}} [\mathrm{i}\boldsymbol{p} \cdot \mathrm{i}\boldsymbol{p}'] \left[ b_{-\boldsymbol{p}} + a_{\boldsymbol{p}}^\dagger \right] \left[ a_{-\boldsymbol{p}'} + b_{\boldsymbol{p}'}^\dagger \right] \\
+ m^2 \frac{1}{2\sqrt{\omega_{\boldsymbol{p}} \omega_{\boldsymbol{p}'}}} \left[ b_{-\boldsymbol{p}} + a_{\boldsymbol{p}}^\dagger \right] \left[ a_{-\boldsymbol{p}'} + b_{\boldsymbol{p}'}^\dagger \right] \Bigg]
\end{aligned} \tag{38}$$

Now the second and third summand share a lot and I can factor that out again.

$$\begin{aligned}
= \int \mathrm{d}^3 x \, \frac{\mathrm{d}^3 p \, \mathrm{d}^3 p'}{[2\pi]^6} \exp\left(\mathrm{i}[\boldsymbol{p}+\boldsymbol{p}']\cdot\boldsymbol{x}\right) \Bigg[ - \frac{\sqrt{\omega_{\boldsymbol{p}} \omega_{\boldsymbol{p}'}}}{2} \left[ b_{-\boldsymbol{p}} - a_{\boldsymbol{p}}^\dagger \right] \left[ a_{-\boldsymbol{p}'} - b_{\boldsymbol{p}'}^\dagger \right] \\
+ \frac{-\boldsymbol{p} \cdot \boldsymbol{p}' + m^2}{2\sqrt{\omega_{\boldsymbol{p}} \omega_{\boldsymbol{p}'}}} \left[ b_{-\boldsymbol{p}} + a_{\boldsymbol{p}}^\dagger \right] \left[ a_{-\boldsymbol{p}'} + b_{\boldsymbol{p}'}^\dagger \right] \Bigg]
\end{aligned} \tag{39}$$

Now that looks a lot like the formula by Peskin and Schroeder (1995, (2.31)), just that I have a factor 2 more then they do. I am not sure how I or they got that, but in case something is off by a factor of two later on, this is where it started. Maybe it is the factor 1/2 they have in their Lagrange density of the real Klein-Gordon field. There is only one function that depends on $\boldsymbol{x}$, and that is the exponential. That integration will give me a $\delta(\boldsymbol{p}+\boldsymbol{p}')$. I can then carry out the integration over $\boldsymbol{p}'$ to eliminate it by applying $\boldsymbol{p}' \to -\boldsymbol{p}$.

$$= \int \frac{\mathrm{d}^3 p}{[2\pi]^3} \left[ -\frac{\omega_{\boldsymbol{p}}}{2} \left[ b_{-\boldsymbol{p}} - a_{\boldsymbol{p}}^\dagger \right] \left[ a_{\boldsymbol{p}} - b_{-\boldsymbol{p}}^\dagger \right] + \frac{p^2+m^2}{2\omega_{\boldsymbol{p}}} \left[ b_{-\boldsymbol{p}} + a_{\boldsymbol{p}}^\dagger \right] \left[ a_{\boldsymbol{p}} + b_{-\boldsymbol{p}}^\dagger \right] \right] \tag{40}$$

The definition was $\omega_{\boldsymbol{p}}^2 = p^2 + m^2$. The fraction will then simplify and both of the fractions can be factored out. I change the order of the two summands to get rid of that additional minus sign.

$$= \int \frac{\mathrm{d}^3 p}{[2\pi]^3} \frac{\omega_{\boldsymbol{p}}}{2} \left[ \left[ b_{-\boldsymbol{p}} + a_{\boldsymbol{p}}^\dagger \right] \left[ a_{\boldsymbol{p}} + b_{-\boldsymbol{p}}^\dagger \right] - \left[ b_{-\boldsymbol{p}} - a_{\boldsymbol{p}}^\dagger \right] \left[ a_{\boldsymbol{p}} - b_{-\boldsymbol{p}}^\dagger \right] \right] \tag{41}$$

Since I do not see a shorter way, I expand all those products of the ladder operators. While doing that, I am careful not to commute anything.

$$= \int \frac{\mathrm{d}^3 p}{[2\pi]^3} \frac{\omega_{\boldsymbol{p}}}{2} \left[ b_{-\boldsymbol{p}} a_{\boldsymbol{p}} + b_{-\boldsymbol{p}} b_{-\boldsymbol{p}}^\dagger + a_{\boldsymbol{p}}^\dagger a_{\boldsymbol{p}} + a_{\boldsymbol{p}}^\dagger b_{-\boldsymbol{p}}^\dagger - \left[ b_{-\boldsymbol{p}} a_{\boldsymbol{p}} - b_{-\boldsymbol{p}} b_{-\boldsymbol{p}}^\dagger - a_{\boldsymbol{p}}^\dagger a_{\boldsymbol{p}} + a_{\boldsymbol{p}}^\dagger b_{-\boldsymbol{p}}^\dagger \right] \right] \tag{42}$$

I factor out the sign and see about the remaining ones.

$$= \int \frac{\mathrm{d}^3 p}{[2\pi]^3} \omega_{\boldsymbol{p}} \left[ b_{-\boldsymbol{p}} b_{-\boldsymbol{p}}^\dagger + a_{\boldsymbol{p}}^\dagger a_{\boldsymbol{p}} \right] \tag{43}$$

I am not sure about this, but I think I can remove the minus sign taking the hermitian conjugate of the ladder operators $b$.

$$= \int \frac{\mathrm{d}^3 p}{[2\pi]^3} \omega_{\boldsymbol{p}} \left[ a_{\boldsymbol{p}}^\dagger a_{\boldsymbol{p}} + b_{\boldsymbol{p}}^\dagger b_{\boldsymbol{p}} \right] \tag{44}$$

Now the whole thing is symmetric in $a$ and $b$ which is nice. There is no zero point energy like the real Klein-Gordan field seems to have (ibid., (2.31)).

**Two sets of particles** The Hamiltonian contains two sets of ladder operators, $a$ and $b$. Those independently create particles with the same momentum $\boldsymbol{p}$, mass $m$ and energy $\omega_{\boldsymbol{p}}$.

### 1.3 Symmetry

**Global symmetry** In all terms only $X^* X = |X|^2$ comes up, where $X = \phi, \dot{\phi}, \pi$. A constant phase factor would cancel out in those and therefore not change the action, Lagrangian or the Hamiltonian. The field and its derivatives and therefore the coordinates would change. But the equations of motions would retain their form.

This can also be shown explicitly using Noether's theorem. The infinitesimal version of the transformation is given by

$$\phi \to \tilde{\phi} = \phi + \mathrm{i}\alpha\phi + \mathrm{O}(\alpha^2). \tag{45}$$

The Lagrangian density then transforms like this:

$$\mathscr{L} = \phi^*_{,\mu}\phi^{,\mu} - m^2\phi^*\phi \tag{46}$$

Now I replace $\phi$ with $\tilde{\phi}$.

$$= [1+\mathrm{i}\alpha]\tilde{\phi}^*_{,\mu}[1-\mathrm{i}\alpha]\tilde{\phi}^{,\mu} - m^2[1+\mathrm{i}\alpha]\phi^*[1-\mathrm{i}\alpha]\phi + \mathrm{O}(\alpha^2) \tag{47}$$

$$= [1+\alpha^2]\tilde{\phi}^*_{,\mu}\tilde{\phi}^{,\mu} - m^2[1+\alpha^2]\phi^*\phi + \mathrm{O}(\alpha^2) \tag{48}$$

The $\alpha^2$ terms can be put into the Landau symbol like this:

$$= \tilde{\phi}^*_{,\mu}\tilde{\phi}^{,\mu} - m^2\phi^*\phi + \mathrm{O}(\alpha^2) \tag{49}$$

The Lagrangian density has not changed in first order, just like argued in the in the first paragraph.

**Conserved quantity** Using this, I can compute the conserved current. Since the Lagrangian density is unchanged, $\mathscr{J} = 0$ (that is a script $J$). The conserved current $j$ then is:

$$j^\mu = \frac{\partial \mathscr{L}}{\partial \phi_{,\mu}}\mathrm{i}\phi - \frac{\partial \mathscr{L}}{\partial \phi^*_{,\mu}}\mathrm{i}\phi^* = \mathrm{i}\left[\frac{\partial \mathscr{L}}{\partial \phi_{,\mu}}\phi - \frac{\partial \mathscr{L}}{\partial \phi^*_{,\mu}}\phi^*\right] \tag{50}$$

There are two terms because I have to independent fields that I have to take account. The minus comes from the complex conjugation of the i.

**Side question**

Is $j$ or $\mathscr{J}$ the Noether current?

**Conserved charge** The zeroth component of the conserved current density, $j^0$ is the charge density. The spatial integral over this charge is the total charge:

$$Q = \int \mathrm{d}^3 x \, j^0. \tag{51}$$

Here the charge density is given by:

$$j^\mu = \mathrm{i} \left[ \frac{\partial \mathscr{L}}{\partial \dot{\phi}} \phi - \frac{\partial \mathscr{L}}{\partial \dot{\phi}^*} \phi^* \right] = \mathrm{i} \left[ \pi \phi - \pi^* \phi^* \right] \tag{52}$$

From this, the conserved charge is

$$Q = \mathrm{i} \int \mathrm{d}^3 x \left[ \pi \phi - \pi^* \phi^* \right], \tag{53}$$

which differs by a factor of $-2$ from Equation (8) on the problem set. Also the order of $\pi$ and $\phi$ is different. Those two do not commute, both would give a $\mathrm{i}\delta(x)$, except that the complex conjugates would give a $-\mathrm{i}\delta(x)$. Those do not cancel.

**Side question**

How do I derive the conserved charge given in Equation (8) on the problem set?

### 1.4 Conserved charge

For some reason my conserved charge, Equation (53), differs from the one given on the problem set. I will continue to use the one from the problem set. The conserved charge given is

$$Q = \int d^3x \, \frac{i}{2} [\phi^* \pi^* - \pi \phi] \tag{54}$$

Now I can insert the expansions of the field and momentum density in terms of ladder operators from Equations (31), (32), (33) and (34).

$$\begin{aligned} &= \frac{1}{2} \int d^3x \int \frac{d^3p \, d^3p'}{[2\pi]^6} \sqrt{\frac{\omega_{\boldsymbol{p}'}}{\omega_{\boldsymbol{p}}}} \exp\left(i[\boldsymbol{p} + \boldsymbol{p}'] \cdot \boldsymbol{x}\right) \\ &\quad \times \left[\left[b_{-\boldsymbol{p}} + a^\dagger_{\boldsymbol{p}}\right]\left[b_{-\boldsymbol{p}'} - a^\dagger_{\boldsymbol{p}'}\right] - \left[a_{-\boldsymbol{p}'} - b^\dagger_{\boldsymbol{p}'}\right]\left[a_{-\boldsymbol{p}} + b^\dagger_{\boldsymbol{p}}\right]\right] \end{aligned} \tag{55}$$

The integration over $\boldsymbol{x}$ again yields a $\delta(\boldsymbol{p} + \boldsymbol{p}')$-distribution that sets $\boldsymbol{p}' := -\boldsymbol{p}$.

$$= \frac{1}{2} \int \frac{d^3p}{[2\pi]^3} \left[\left[b_{-\boldsymbol{p}} + a^\dagger_{\boldsymbol{p}}\right]\left[b_{\boldsymbol{p}} - a^\dagger_{-\boldsymbol{p}}\right] - \left[a_{\boldsymbol{p}} - b^\dagger_{-\boldsymbol{p}}\right]\left[a_{-\boldsymbol{p}} + b^\dagger_{\boldsymbol{p}}\right]\right] \tag{56}$$

Now I have to compute the products of all that again, just like in the derivation of the Hamiltonian.

$$= \frac{1}{2} \int \frac{\mathrm{d}^3 p}{[2\pi]^3} \left[ b_{-p} b_p - b_{-p} a_{-p}^\dagger + a_p^\dagger b_p - a_p^\dagger a_{-p}^\dagger - \left[ a_p a_{-p} + a_p b_p^\dagger - b_{-p}^\dagger a_{-p} - b_{-p}^\dagger b_p^\dagger \right] \right] \tag{57}$$

I factor out the minus sign.

$$= \frac{1}{2} \int \frac{\mathrm{d}^3 p}{[2\pi]^3} \left[ b_{-p} b_p - b_{-p} a_{-p}^\dagger + a_p^\dagger b_p - a_p^\dagger a_{-p}^\dagger - a_p a_{-p} - a_p b_p^\dagger + b_{-p}^\dagger a_{-p} + b_{-p}^\dagger b_p^\dagger \right] \tag{58}$$

I sort the terms and commute $a$ with $b$.

$$= \frac{1}{2} \int \frac{\mathrm{d}^3 p}{[2\pi]^3} \left[ -a_p a_{-p} - a_p b_p^\dagger - a_p^\dagger a_{-p}^\dagger + a_p^\dagger b_p + a_{-p} b_{-p}^\dagger - a_{-p}^\dagger b_{-p} + b_{-p}^\dagger b_p^\dagger + b_{-p} b_p \right] \tag{59}$$

The bounds of the integration is symmetric, so I can move the minus sign in the $\boldsymbol{p}$ to the last term or remove it completely.

$$= \frac{1}{2} \int \frac{\mathrm{d}^3 p}{[2\pi]^3} \left[ -a_p a_{-p} - a_p^\dagger a_{-p}^\dagger + a_p^\dagger b_p - a_p^\dagger b_p + a_p b_p^\dagger - a_p b_p^\dagger + b_p^\dagger b_{-p}^\dagger + b_p b_{-p} \right] \tag{60}$$

The mixed term vanish now.

$$= \frac{1}{2} \int \frac{\mathrm{d}^3 p}{[2\pi]^3} \left[ -a_p a_{-p} - a_p^\dagger a_{-p}^\dagger + b_p^\dagger b_{-p}^\dagger + b_p b_{-p} \right] \tag{61}$$

Now there is nothing I directly see to go on. However, if $a_{-p} = a_p^\dagger$, then I could further simplify it. It makes sense somewhat: Creating a particle with opposite wavenumber is the same as removing one particle with the actual wavenumber. So I can do this?

$$= \frac{1}{2} \int \frac{\mathrm{d}^3 p}{[2\pi]^3} \left[ -a_p a_p^\dagger - a_p^\dagger a_p + b_p^\dagger b_p + b_p b_p^\dagger \right] \tag{62}$$

I shifted the minus sign in the momentum from one operator to the other with the argument of the symmetric integration bounds. If I had shifted them in another way around before, I would now be done and have

$$= \int \frac{\mathrm{d}^3 p}{[2\pi]^3} \left[ -n_{ap} + n_{bp} \right]. \tag{63}$$

$n$ is the occupation number operator $a^\dagger a$. That means that the particles $a$ have charge $-1$ and the particles $b$ have charge 1. I could have chosen the signs differently and arrived at the charges the other way around as well.

### 1.5 Two fields

**Symmetries of two fields** So far I have been dealing with a single (yet complex) field and a symmetry operation which just had one generator. Both is going to change now. The Lagrangian is now composed of parts for both fields $\phi_a$ with $a = 1, 2$:

$$\mathscr{L} = \sum_a \mathscr{L}_a = \phi^*_{a,\mu} \phi^{a,\mu} - m^2 \phi^*_a \phi^a. \tag{64}$$

**Side question**

Now $\boldsymbol{\phi}$ seems to be a two-vector here since it consists of two fields. Is *that* a spinor?

If I look at the fields individually, I only have to symmetry operation with parameter $\alpha$. Once I thought about $\boldsymbol{\phi}$ as a vector, it came natural to look at transformation matrices:

$$\tilde{\boldsymbol{\phi}} = \boldsymbol{A} \cdot \boldsymbol{\phi}. \tag{65}$$

The Lagrangian density now looks like this:

$$\mathscr{L} = \left[\boldsymbol{A}^{-1} \cdot \tilde{\boldsymbol{\phi}}_{,\mu}\right]^\dagger \cdot \left[\boldsymbol{A}^{-1} \cdot \tilde{\boldsymbol{\phi}}^{,\mu}\right] - m^2 \left[\boldsymbol{A}^{-1} \cdot \tilde{\boldsymbol{\phi}}\right]^\dagger \cdot \left[\boldsymbol{A}^{-1} \cdot \tilde{\boldsymbol{\phi}}\right]. \tag{66}$$

Now I can apply the hermitian conjugate to the square brackets. In the next step it will become clear that $\boldsymbol{A}$ has to be unitary. I will therefore say that the symmetry transformations have to be SU(2) matrices. Then the adjoint of the inverse is the matrix itself.

$$= \tilde{\boldsymbol{\phi}}^\dagger_{,\mu} \cdot \boldsymbol{A} \cdot \boldsymbol{A}^{-1} \cdot \tilde{\boldsymbol{\phi}}^{,\mu} - m^2 \tilde{\boldsymbol{\phi}}^\dagger \cdot \boldsymbol{A} \cdot \boldsymbol{A}^{-1} \cdot \tilde{\boldsymbol{\phi}} \tag{67}$$

The matrices cancel out and one is left with the unaltered Lagrangian density.

$$= \tilde{\boldsymbol{\phi}}^{\dagger}_{,\mu} \cdot \tilde{\boldsymbol{\phi}}^{,\mu} - m^2 \tilde{\boldsymbol{\phi}}^{\dagger} \cdot \tilde{\boldsymbol{\phi}} \tag{68}$$

Note that the $\boldsymbol{\phi}$ are field vectors (in bold) and that the contraction is implied.

I have now found that any SU(2) matrix transforms the field vector in a way that leaves the equations of motion invariant. There is also the U(1) symmetry that is generated by $\exp(\mathrm{i}\alpha)\boldsymbol{1}_2$ where $\boldsymbol{1}_2$ is the unit matrix in two dimensions. From here I can go to Noether's theorem and compute the conserved quantities.

There are four generators $T_\lambda$ here, one from U(1) and three from SU(2). The generators are in the Physicist's convention of hermitian generators:

$$T_0 = \boldsymbol{1}_2, \quad T_i = \sigma_i \tag{69}$$

According to Wikipedia (2014) it is customary to include the unit matrix $\boldsymbol{1}_2$ as a zeroth Pauli matrix. I do see why from this. $\boldsymbol{1}_2$ is the one and only generator of U(1), but expressed in two dimensions. This is a reducible representation of U(1), but I want to operate on two-dimensional quantities, so I need a representation in this number of dimensions. And so I just take the simple generator 1 twice and have $\boldsymbol{1}_2 = \boldsymbol{1}_1 \oplus \boldsymbol{1}_1$ as a generator in two dimensions.

**Conserved quantities** Using the generators I can now express the four basic unitary matrices that give me symmetry transformations as

$$\boldsymbol{A} = \exp\left(\mathrm{i}\alpha^\lambda T_\lambda\right) \tag{70}$$

where $\boldsymbol{\alpha}$ is now a parameter four-vector. I can then compute the Noether charge densities from that:

$$j^0_\lambda = \frac{\partial \mathscr{L}}{\partial \dot{\phi}^a} \mathrm{i} [\boldsymbol{\sigma}_\lambda \cdot \boldsymbol{\phi}]^a + \text{h.c.}, \tag{71}$$

where h.c. stands for the hermitian conjugate of all the previous terms in the same scope. The "·" stands for a single contraction of indices. I can now compute the derivative and replace it by the canonical momentum density.

$$= \pi_a \mathrm{i} [\boldsymbol{\sigma}_\lambda \cdot \boldsymbol{\phi}]^a + \text{h.c.}. \tag{72}$$

Then I can write this a whole vector-matrix-vector expression.

$$= \mathrm{i}\boldsymbol{\pi} \cdot \boldsymbol{\sigma}_\lambda \cdot \boldsymbol{\phi} + \text{h.c.} \tag{73}$$

The conserved charges are now given by:

$$Q_\lambda = \mathrm{i} \int \mathrm{d}^3x \, \boldsymbol{\pi} \cdot \boldsymbol{\sigma}_\lambda \cdot \boldsymbol{\phi} + \text{h.c.} \tag{74}$$

This again differs by a factor −2 from the result given on the problem set. The change in ordering of $\pi$ and $\phi$ is accounted by the hermitian conjugation instead of the plain complex conjugation.

**Commutation relations** The commutation relations of the $Q_i$ follow now. I already use the linearity of the integral to make it a bit shorter.

$$[Q^i, Q^j] = \mathrm{i} \int \mathrm{d}^3x\, [\boldsymbol{\pi} \cdot \boldsymbol{\sigma}^i \cdot \boldsymbol{\phi}, \boldsymbol{\pi} \cdot \boldsymbol{\sigma}^j \cdot \boldsymbol{\phi}] + \text{h.c.} \tag{75}$$

I have to expand this with indices again to be able to commute some of the parts. All the field indices go to the bottom to make it a bit easier on the eye. Summation convention is still applicable.

$$= \mathrm{i} \int \mathrm{d}^3x\, [\pi_a \sigma^i_{ab} \phi_b, \pi_c \sigma^j_{cd} \phi_d] + \text{h.c.} \tag{76}$$

Now the Pauli matrices are just numbers and I can commute them with $\pi$ and $\phi$. I cannot commute the latter with each other since the canonical commutation relations apply for those. But I do not need to commute those anyway.

$$= \mathrm{i} \int \mathrm{d}^3x\, [\pi_a \phi_b \sigma^i_{ab}, \sigma^j_{cd} \pi_c \phi_d] + \text{h.c.} \tag{77}$$

The parts apart of the Pauli matrices can be pulled out the commutator since they were going to commute with those matrices anyway. This leaves me with the commutator of the Pauli matrices themselves.

$$= \mathrm{i} \int \mathrm{d}^3x\, \pi_a \phi_b [\sigma^i_{ab}, \sigma^j_{cd}] \pi_c \phi_d + \text{h.c.} \tag{78}$$

Therefore, the $Q_i$ commute just like the $\sigma_i$.

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