QFT Problem Set 12

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I had a hard time understanding this last week. The tutorial cleared it up, so I want to try this again. The exam will contain one problem with a loop.

Part 2

We need to compute

$$\int \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2 - A)^n} .$$

The coordinates are not on equal footing. We want to use spherical coordinates and should get to a metric tensor with signature ± 4, not ± 2 as we have with SR.

Therefore do a Wick rotation and introduce Euclidian components.

$$p^0 = i p_E^0 \qquad \vec{p} = \vec{p}_E$$

$$p^2 \underset{\substack{\uparrow \\ \text{Minkowski scalar} \\ \text{product}}}{=} (p^0)^2 - \vec{p}^2 \underset{\substack{\uparrow \\ \text{Wick} \\ \text{rotation}}}{\longmapsto} (ip^0)^2 - \vec{p}^2 = -(p^0)^2 - \vec{p}^2 \underset{\substack{\uparrow \\ \text{Identification} \\ \text{with Euclidian version}}}{=} -p_E^2$$

We then obtain:

$$\int \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2-A)^n} \longmapsto \int \frac{i\, d^d p_E}{(2\pi)^d} \frac{1}{(-p_E^2-A)^n}$$

$$= (-1)^n i \int \frac{d^d p_E}{(2\pi)^d} \frac{1}{(p_E^2+A)^n}$$

Now we are in Euklidian space-time with signature +4. Time and space are on equal footing from a geometric perspective now.

Generalized spherical coordinates are given via the following measure:

$$d^d p_E = P^{d-1}\, dP\, d\varphi \prod_{k=1}^{d-2} \sin(\vartheta_k)^k\, d\vartheta_k.$$

Capital P is the radial component in this momentum space. The "p"s appearing next will be the radius only. I will not write the capital "P" that explicit in the following.

The integral now is:

$$(-1)^n i \int \frac{1}{(2\pi)^d} P^{d-1}\, dP\, d\varphi \prod_{k=1}^{d-2} \sin(\vartheta_k)^k\, d\vartheta_k \frac{1}{(P^2+A)^n}$$

The radial and angular part can be separated into factors. I will focus on the angular part first. This was not too hard, see the thing that I have in.

Reorded I have:

$$\underbrace{\frac{1}{(2\pi)^d}\int d\varphi \prod_{k=1}^{d-2} \sin(\vartheta_k)^k \, d\vartheta_k}_{\text{Angular part}} \; \underbrace{(-1)^n i \, P^{d-1} \, dP \frac{1}{(P^2+A)^n}}_{\text{Radial part}}$$

As one might have seen before, the angular part is the surface area of a sphere embedded in d dimensions. The actual surface has d-1 dimensions. The result is given on my previous work, here it is:

$$\int d\Omega_d = \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}$$

The integral then becomes:

$$\frac{(-1)^n i}{(2\pi)^d} \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)} \int_0^{\infty} dP \frac{P^{d-1}}{(P^2+A)^n}$$

I will ignore the factors in the front for now. The A should be extracted outside of the integral. A can be extracted like so.

$$\frac{1}{(P^2+A)^n} = \frac{1}{A^n} \frac{1}{\left(\left(\frac{P}{\sqrt{A}}\right)^2+1\right)^n}$$

Then the integral is

$$A^{-n}\int_0^\infty dP \, \frac{P^{d-1}}{\left(\left(\frac{P}{\sqrt{A}}\right)^2+1\right)^n}.$$

Next I need to substitute.

$$\rho := \frac{P^2}{A} \qquad d\rho = \frac{2P\,dP}{A} \qquad P = \sqrt{A\rho}$$

$$A^{-n}\int_0^\infty d\rho \, \frac{A}{2\sqrt{\rho A}} \; \frac{A^{\frac{d-1}{2}}\,\rho^{\frac{d-1}{2}}}{(\rho+1)^n}$$

$$= \frac{1}{2} A^{-n+1-\frac{1}{2}+\frac{d-1}{2}} \int_0^\infty d\rho \, \frac{\rho^{\frac{d-1}{2}-\frac{1}{2}}}{(\rho+1)^n}$$

$$= \frac{1}{2} A^{\frac{d}{2}-n} \int_0^\infty d\rho \, \frac{\rho^{\frac{d}{2}-1}}{(\rho+1)^n}$$

This can be rewritten just slightly:

$$= \frac{1}{2} A^{\frac{d}{2}-n} \int_0^\infty d\rho \, \rho^{\frac{d}{2}-1} (\rho+1)^{-n}$$

On the problem set, Equation (7) gives the Beta function:

$$B(a+1, b+1) = \int_0^\infty dt \; t^a \, [t+1]^{-2-a-b}.$$

In my case I have

$$a = \frac{d}{2} - 1 \qquad -2 - a - b = -n$$

$$2 + \frac{d}{2} - 1 + b = n$$

$$b = n - \frac{d}{2} - 1$$

Hence:

$$\frac{1}{2} A^{\frac{d}{2}-n} \int_0^\infty d\rho \, \rho^{\frac{d}{2}-1} (\rho+1)^{-n} = \frac{1}{2} A^{\frac{d}{2}-n} B\left(\frac{d}{2}, n-\frac{d}{2}\right).$$

The explicit form of the Beta function was also given there in terms of Γ functions.

$$\frac{1}{2} A^{\frac{d}{2}-n} B\left(\frac{d}{2}, n-\frac{d}{2}\right) = \frac{1}{2} A^{\frac{d}{2}-n} \frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(n-\frac{d}{2}\right)}{\Gamma(n)}$$

Next I need to join this with the angular part.

$$\frac{(-1)^n i}{(2\pi)^d} \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)} \frac{1}{2} A^{\frac{d}{2}-n} \frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(n-\frac{d}{2}\right)}{\Gamma(n)}$$

$$= \frac{(-1)^n i}{2^d \pi^{d/2}} \frac{\Gamma\left(n-\frac{d}{2}\right)}{\Gamma(n)} A^{\frac{d}{2}-n}$$

$$2^d \pi^{d/2} = \sqrt{4}^d \pi^{d/2} = (4\pi)^{d/2}$$

This matches the result given on the problem set.

Good, so this became clear in the tutorial. Thank you!

Part 4

The invariant matrix element can be read off:

$$i\mathcal{M} = \frac{1}{2}\lambda\mu^{2[2-\omega]} \int \frac{d^{2\omega}p}{(2\pi)^{2\omega}} \frac{1}{p^2 - m^2}$$

I did that in the version that I handed in in more detail. Here I can just use the formula that I have just derived before for $d = 2\omega$, $A = m^2$, $n = 1$

$$i\mathcal{M} = \frac{1}{2}\lambda\mu^{2[2-\omega]} \int \frac{d^{2\omega}p}{(2\pi)^{2\omega}} \frac{1}{p^2 - m^2}$$

$$= \frac{1}{2}\lambda\mu^{2[2-\omega]} \frac{-i}{(4\pi)^{\omega}} \frac{\Gamma(1-\omega)}{\Gamma(1)} m^{2\omega-2}$$

$\Gamma(1) = 1$

It seemed handy to group the terms which are raised to the ω^{th} power in one term.

$$= -\frac{i}{2}\lambda\mu^4\mu^{-2\omega}\frac{1}{(4\pi)^{\omega}} m^{2\omega} m^{-2} \Gamma(1-\omega)$$

$$= -\frac{i}{2}\frac{\lambda\mu^4}{m^2}\left(\frac{m^2}{4\pi\mu^2}\right)^{\omega} \Gamma(1-\omega)$$

Then I can use the expansion given in Equation (9). Also one needs

$$a^b = \exp(\ln(a)\cdot b) = 1 + \ln(a)\, b + \mathcal{O}(b^2).$$

The expansion is to be around $\omega = 2$. Therefore it makes sense to define: $\varepsilon = \omega - 2$.

$$= \frac{-i}{2} \frac{\lambda \mu^4}{m^2} \left(\frac{m^2}{4\pi\mu^2} \right)^{2+\varepsilon} \Gamma(-1+\varepsilon)$$

$$-\left[\frac{1}{\varepsilon} + \psi(2) + O(\varepsilon) \right]$$

$$-\frac{i\lambda m^2}{32\pi^2} \qquad \left(\frac{m^2}{4\pi\mu^2} \right)^2 \qquad \left(\frac{m^2}{4\pi\mu^2} \right)^{\varepsilon} \to 1 + \ln\left(\frac{m^2}{4\pi\mu^2} \right) \varepsilon$$

Everything together:

$$\frac{i\lambda m^2}{32\pi^2} \left[1 + \ln\left(\frac{m^2}{4\pi\mu^2} \right) \varepsilon \right] \left[\frac{1}{\varepsilon} + \psi(2) + O(\varepsilon) \right]$$

This can be factored out and have terms of $O(\varepsilon)$ dropped.

$$= \frac{i\lambda m^2}{32\pi^2} \left[\frac{1}{\varepsilon} + \ln\left(\frac{m^2}{4\pi\mu^2} \right) + \psi(2) + O(\varepsilon) \right]$$

I do not recall the correct result from class. There was some divergence in the logarithm as well, I think. Please be a bit careful with this unreviewed result.