Disclaimer

This is a problem set (as turned in) for the module physics754.

This problem set is not reviewed by a tutor. This is just what I have turned in.

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[disclaimer]

physics754 – Problem Set 10

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In the energy momentum tensor $\boldsymbol{T}$*, the electromagnetic waves are a form of energy density. With* $\boldsymbol{G} = 8\pi G\boldsymbol{T}$*, this means that photons create curvature.*

The gravitational waves carry energy. Are they also a sort of energy density? If so, do they need to be included in the energy momentum tensor or are they taken care of by the Einstein tensor itself?

P.11: Strong gravitational waves

(a)

Lower the index on $\boldsymbol{A}$ and form $\boldsymbol{A} \otimes \boldsymbol{A}$ with it. It will have the needed form.

(b): Pull back

I compute the Jacobian:

$$\frac{\partial x^\mu}{\partial y^\nu} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^\mu_{\ \nu}$$

With that, I can compute the elements of the transformed metric. That worked out, so I do not write it down here explicitly.

(c): Inverse metric

Now one has to compute the inverse metric. The metric is given by

$$\hat{g}_{\mu\nu} = \begin{pmatrix} 2f & 1 \\ 1 & 0 \end{pmatrix}^\mu_{\ \nu}.$$

The inverse can be computed easily with the gauss algorithm. Start with $(\hat{\boldsymbol{g}}|\mathbb{1})$ and bring the first part to the identity. Then the system hat the form $(\mathbb{1}|\hat{\boldsymbol{g}}^{-1})$. The inverse is

$$\hat{g}^{-1}_{\mu\nu} = \begin{pmatrix} 0 & 1 \\ 1 & -2f \end{pmatrix}^\mu_{\ \nu}.$$

Part (d): Christoffel symbols

The only nonzero derivatives are the $g_{00,\delta}$, such that the Christoffel symbols only consist of f and $f_{,\delta}$. I did the first, and I will not write them down here.

Part (e): Additional condition

The

$$A^\mu f_{,\mu} = 0$$

means

$$[\partial_0 + \partial_1] f = 0$$

when written out. This equation can be differentiated with respect to x^0 and x^1, which will yield

$$[\partial_0^2 + \partial_0 \partial_1] f = 0, \quad [\partial_0 \partial_1 + \partial_1^2] f = 0.$$

I then subtract those two equations giving me

$$[\partial_0^2 - \partial_1^2] f = 0.$$

Subtracting this from $\Box f = 0$ leaves the required

$$[\partial_2^3 + \partial_3^2] f = 0.$$

Part (f): Meets conditions

f meets the homogeneous wave equation because of the minus sign. Since it does not depend on the first two coordinates, it also meets $A^\mu f_{,\mu} = 0$ where $\boldsymbol{A}$ is nonzero only for the first two components.

H.13: Geodesics

A general geodesic equation is given:

$$\frac{\mathrm{d}}{\mathrm{d}\tau} g_{\mu\nu} \dot{x}^\nu = \frac{1}{2} \partial_\mu g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta.$$

The problem asks for solutions $\boldsymbol{x}(\tau)$ of this equation with $\|\boldsymbol{x}\| = 1$ or $\|\boldsymbol{x}\| = 0$. That means that the $g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta$ is a constant, and the complete right hand side of the equation is zero. This leaves

$$\frac{\mathrm{d}}{\mathrm{d}\tau} g_{\mu\nu} \dot{x}^\nu = 0.$$

The metric is given by its nonzero components

$$g_{00} = 1 + f, \quad g_{10} = g_{01} = -f, \quad g_{11} = -1 + f, \quad g_{22} = g_{33} = -1$$

where f is a function of x^2 and x^3 only, namely

$$f(x^2, x^3) = [x^2]^2 - [x^3]^3.$$

Using product and chain rule, the differential equation becomes

$$g_{\mu\nu,\lambda} x^\lambda \dot{x}^\nu + g_{\mu\nu} \ddot{x}^\nu = 0.$$

The condition $\|\boldsymbol{x}\| = c$ can be written as

$$[1+f]\dot{x}^0 \dot{x}^0 - 2f \dot{x}^0 \dot{x}^1 + [-1+f]\dot{x}^1 \dot{x}^1 - [\dot{x}^2]^2 - [\dot{x}^2]^2 = c.$$

I have tried to transform the equation to the coordinates y, but that would require the metric $\hat{\boldsymbol{g}}$ to be

$$\hat{g}_{00} = \frac{1-f}{2}, \quad \hat{g}_{10} = \hat{g}_{01} = \frac{1+f}{2}, \quad \hat{g}_{11} = \frac{1+3f}{2}, \quad \hat{g}_{22} = \hat{g}_{33} = -1$$

in order to match the $\|\boldsymbol{y}\| = c$ in those coordinates as well. Therefore, I have continued to do this problem int the x coordinates.

I then put the metric $\boldsymbol{g}$ into the equation that I am supposed to solve. Since they are actually four equations, I start with $\mu = 0$. The derivatives of the metric are the derivatives of f, and those are rather simple with respect to any coordinate. Only the derivatives with respect to the components 2 and 3 are nonzero. And μ and ν have to be either 0 or 1. Therefore, the only nonzero summands are

$$g_{00,2}\dot{x}^2\dot{x}^0 + g_{01,2}\dot{x}^2\dot{x}^1 + g_{00,3}\dot{x}^3\dot{x}^0 + g_{01,3}\dot{x}^3\dot{x}^1 + g_{0\nu}\ddot{x}^\nu = 0.$$

I can then expand the occurrences of the metric and yield

$$2x^2\dot{x}^2\dot{x}^- - 2x^2\dot{x}^2\dot{x}^1 - 2x^3\dot{x}^3\dot{x}^0 + 2x^3\dot{x}^3\dot{x}^1 + [1-f]\ddot{x}^0 - f\ddot{x}^1 = 0.$$

Observing that

$$2x^\mu\dot{x}^\mu = \frac{\mathrm{d}}{\mathrm{d}\tau}[x^\mu]^2$$

the previous equation can be written terser as

$$\dot{f}[\dot{x}^0 + \dot{x}^1] + [1+f]\ddot{x}^0 - f\ddot{x}^1 = 0.$$

$\mu = 1$ gives a similar equation:

$$\dot{f}[-\dot{x}^0 + \dot{x}^1] - f\ddot{x}^0 + [1-f]\ddot{x}^1 = 0.$$

The last two equations are simple, they are $\ddot{x}^2 = 0$ and $\ddot{x}^3 = 0$. Their solutions are just linear functions in τ:

$$x^2(\tau) = c_1 + c_2\tau, \quad x^3(\tau) = c_3 + c_4\tau.$$

That gives a system of coupled, second order differential equations. I think that they are also linear.

$$\begin{aligned} \dot{f}[\dot{x}^0 + \dot{x}^1] + [1+f]\ddot{x}^0 - f\ddot{x}^1 &= 0 \\ \dot{f}[-\dot{x}^0 + \dot{x}^1] - f\ddot{x}^0 + [1-f]\ddot{x}^1 &= 0 \end{aligned}$$

Since f is a function of x^2 and x^3 only, and those are solved so far, this is just a known function of τ at this point. I have tried to generate two new equations by adding and subtracting the previous two:

$$\begin{aligned} &2\dot{f}\dot{x}^1 + \ddot{x}^0 + [1-2f]\ddot{x}^1 = 0 \\ &2\dot{f}\dot{x}^0 - \ddot{x}^1 + [1+2f]\ddot{x}^0 = 0 \end{aligned}$$

I then tried to use the coordinates y from here. By just substituting the various derivatives of x^μ with y^μ, I was able to transform the two equations to

$$\begin{aligned} \dot{f}\dot{y}^1 + \ddot{y}^1 + f\ddot{y}^0 &= 0 \\ \dot{f}\dot{y}^0 + f\ddot{y}^1 &= 0. \end{aligned}$$

The sum of those two can be written as

$$\frac{\mathrm{d}}{\mathrm{d}\tau} f[\dot{y}^0 + \dot{y}^1] + \ddot{y}^1 = 0$$

where I can drop one derivative and get

$$f[\dot{y}^0 + \dot{y}^1] + \dot{y}^1 = 0.$$

I just do not see where I could continue from here. So far, I have not used $\|\boldsymbol{x}\| = c$, which is probably required to go on. However, I do not see how.