Disclaimer

This is a problem set (as turned in) for the module physics754.

This problem set is not reviewed by a tutor. This is just what I have turned in.

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[disclaimer]

physics754 – Problem Set 9

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P.10: Gravitational Waves

(a) Solution of the equation

The perturbation is given by:

$$h_{\mu\nu}(\boldsymbol{x}) = a_{\mu\nu} \exp(\mathrm{i}\boldsymbol{k}\boldsymbol{x}) + c$$

Then

$$\Box h_{\mu\nu}(\boldsymbol{x}) = -a_{\mu\nu} \exp(\mathrm{i}\boldsymbol{k}\boldsymbol{x}) \left[k_0^2 - |\boldsymbol{k}|^2\right]$$

is 0, since $|\boldsymbol{k}| = 0$. This Ansatz indeed solves (3) from the problem set.

(b)

Equation (7) has to be plugged into (6). The result of that has then to be plugged into (4). (7) into (6) gives me

$$h^{\mu}_{\nu} = \left[\alpha\left[e_1^{\mu} e_{1\nu} - e_2^{\mu} e_{2\nu}\right] + \beta\left[e_1^{\mu} e_{2\nu} + e_2^{\mu} e_{1\nu}\right]\right] \exp(\mathrm{i}\boldsymbol{k}\boldsymbol{x}).$$

I then have to take the derivative with respect to x^λ of the whole thing.

$$h^{\mu}_{\nu,\lambda} = \left[\alpha\left[e_1^{\mu} e_{1\nu} - e_2^{\mu} e_{2\nu}\right] + \beta\left[e_1^{\mu} e_{2\nu} + e_2^{\mu} e_{1\nu}\right]\right] \exp(\mathrm{i}\boldsymbol{k}\boldsymbol{x}) \mathrm{i} k_\lambda$$

With that, I have to show that the equation $h^{\mu}_{\nu,\mu} = h^{\mu}_{\mu,\nu}/2$ holds. So I contract the indices and look at the vectors in full:

$$\alpha\left[\boldsymbol{e}_1\boldsymbol{e}_1\boldsymbol{k} - \boldsymbol{e}_2\boldsymbol{e}_2\boldsymbol{k}\right] + \beta\left[\boldsymbol{e}_2\boldsymbol{e}_1\boldsymbol{k} + \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{k}\right] = \frac{1}{2}\alpha\left[\boldsymbol{k}\boldsymbol{e}_1\boldsymbol{e}_1 - \boldsymbol{k}\boldsymbol{e}_2\boldsymbol{e}_2\right] + \frac{1}{2}\beta\left[\boldsymbol{k}\boldsymbol{e}_1\boldsymbol{e}_2 + \boldsymbol{k}\boldsymbol{e}_2\boldsymbol{e}_1\right].$$

The scalar products are meant to be from right to left, as usual. Now I can use $\boldsymbol{e}_1\boldsymbol{e}_1 = \boldsymbol{e}_2\boldsymbol{e}_2$, $\boldsymbol{e}_1\boldsymbol{e}_2 = 0$ and $k\boldsymbol{e}_i = 0$. With that, both sides are individually zero.

(c) Degrees of freedom

The gauge fixing condition is linear. So I can perform the same steps, yielding

$$\boldsymbol{bkk} + \boldsymbol{kbk} = \frac{1}{2}[\boldsymbol{kkb} + \boldsymbol{kkb}].$$

Since $\boldsymbol{kk} = 0$, this is also true.

$\Box \boldsymbol{h} = 0$ and $\boldsymbol{h}$ being symmetric means that I have 10 equations with 10 integration constants, since it is a second order differential equation. The gauge fixing removes 4 degrees of freedom in those parameters. Then the full family of $\boldsymbol{b}$ removes another four degrees, leaving just two.

(d) Analogy in electrodynamics

Showing that (9) is a solution is fast, since $\Box A_\mu$ produces a $k_0^2 - |\boldsymbol{k}|^2$ which is zero in its nature as a light-like vector.

The linear independence is equivalent to:

$$\forall \boldsymbol{x} \forall \boldsymbol{k} \forall \boldsymbol{k}' \neq \boldsymbol{k} \colon \left(a = b \exp\left(\mathrm{i}[\boldsymbol{k}' - \boldsymbol{k}]\boldsymbol{x}\right) \iff a = 0 \land b = 0\right)$$

And the only solution is that a and b are both zero, I think. So the solutions are linearly independent.

Then, using

$$A_\mu = \left[\alpha e_{1\mu} + \beta e_{2\mu} + \gamma k_\mu\right] \exp(\mathrm{i}\boldsymbol{kx}) + c$$

I can show that this fulfills the Lorenz gauge condition

$$\partial_\mu A^\mu = \left[\alpha e_1^\mu + \beta e_2^\mu + \gamma k^\mu\right] \exp(\mathrm{i}\boldsymbol{kx}) k_\mu$$

since all the scalar products are zero.

The term that is proportional to γ is a gauge transformation since it can be written as the gradient of another function. That function is $-\mathrm{i}\gamma \exp(\mathrm{i}\boldsymbol{kx})$. See the footnote!7 on page 49 in the lecture notes.

Then, the curl of that gradient is zero, so it does not change the magnetic field $\boldsymbol{B}$. The field $\boldsymbol{E}$ is not changed either, since the time dependence removes it again.

H.12: The dust cloud metric on S^3

(a) Pullback of $\boldsymbol{g}$

I will call the space where the $\boldsymbol{x}$ are contained in X. The space of the $\boldsymbol{y}$ will be called Y. Since $r(\psi)$ is given, φ is a mapping that maps $Y \to X$. The associated pullback φ^* therefore operates in the other direction, $X \to Y$.

The Jacobian

$$\frac{\partial x^\mu}{\partial y^\nu}$$

that I will need for the pullback, has diagonal form. The only diagonal entry that is not 1 is the 1, 1 component. That is:

$$r'(\psi) = \frac{1}{\sqrt{a}} \cos(\psi).$$

With that, I can write down metrical tensor that is pulled from X to Y. The metrical tensor and the Jacobian are diagonal, so that the resulting metrical tensor is diagonal as well.

$$\begin{aligned}
\tilde{g}_{00}(\boldsymbol{y}) &= 1 \\
\tilde{g}_{11}(\boldsymbol{y}) &= g_{11}\left(\varphi(\boldsymbol{y})\right) \frac{1}{a} \cos(\psi)^2 \\
&= -\frac{f(t)^2}{a} \\
\tilde{g}_{22}(\boldsymbol{y}) &= -\frac{f(t)^2}{a} \sin(\psi)^2 \\
\tilde{g}_{33}(\boldsymbol{y}) &= -\frac{f(t)^2}{a} \sin(\psi)^2 \sin(\theta)^2
\end{aligned}$$

(b) Vector *n*

The $\tilde{g}_{00} = 1$ was shown before in part (a). For the $\tilde{g}_{ik}$, I will need the Jacobian of $\boldsymbol{n}(t, \psi, \theta, \phi)$. With that, I can write down the components. This is just using trigonometric identities. They give me the results that I had previously, I do not see a point in pretty printing all those calculations here. The last part is easy as well:

$$\frac{\partial n^\alpha}{\partial y^0} = 0 \implies \tilde{g}_{i0} = 0.$$

That leaves the interesting part, the interpretation. The vector $\boldsymbol{n}$ is given as

$$\boldsymbol{n} = \begin{pmatrix} \cos(\psi) \\ \sin(\psi)\sin(\theta)\cos(\phi) \\ \sin(\psi)\sin(\theta)\sin(\phi) \\ \sin(\psi)\cos(\theta) \end{pmatrix},$$

which is a unit vector. Since there are three coordinates, it is a unit vector for all ψ, θ and ϕ and the problem set says something about an S^3, I think that this a S^3 embedded in $\mathbb{E}^4$.

Since metric in $\mathbb{E}^4$ is flat and has an all-positive signature, I can write this sum as

$$\sum_{\alpha=1}^{4} \frac{\partial n^\alpha}{\partial y^i} \frac{\partial n^\alpha}{\partial y^k} = \widehat{\mathrm{D}\boldsymbol{n} \otimes \mathrm{D}\boldsymbol{n}} = \langle \mathrm{D}\boldsymbol{n}, \mathrm{D}\boldsymbol{n} \rangle$$

if you contract over the first upper index of both parts. That again looks like a Gram determinant since they are square matrices:

$$\mathscr{G} = [\mathrm{D}\boldsymbol{n}]^{\mathrm{T}}\,\mathrm{D}\boldsymbol{n}.$$

Since the $\sqrt{\mathscr{G}}$ comes up in the measure (?) on an integral on manifolds, and the $\sqrt{|g|}$ comes up in a lot of our integrals, I assume that they are closely related.

The metric $\tilde{\boldsymbol{g}}$ has the form it has in comoving coordinates. The spatial part seems to have the same metric as a S^3 with a radius being some power or root of $f(t)^2/a$.