Disclaimer

This is a problem set (as turned in) for the module physics754.

This problem set is not reviewed by a tutor. This is just what I have turned in.

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physics754 – Problem Set 6

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Sorry for the landscape printing. The equations got pretty long and I did not want to break them all into several lines since they have so many parentheses.

H.9: Schwarzschild trajectories given by the Jacobi method

I will omit the subscript q since there are no other velocities. Vectors in $\mathbb{R}^3$ are typeset in bold serif italics.

(a) Canonical momenta

The canonical momenta are:

$$p_i = \frac{\partial L}{\partial v^i} = -m \frac{1}{2f(\boldsymbol{q}, \boldsymbol{v}, q^0)} \left[2g_{0i}(q^0, \boldsymbol{q}) + 2g_{ij}(q^0, \boldsymbol{q}) v^j \right].$$

I will assemble the hamiltonian from that. Since the function arguments to do change within the following calculations, I will just omit them. f and g are still functions.

$$
\begin{aligned}
H(\boldsymbol{q},\boldsymbol{p}) &= \left[p_i v^i - L\right]_{v=v(\boldsymbol{q},\boldsymbol{p})} \\
&= -\left[\frac{m}{2f}\left[2g_{0i} + 2g_{ij}v^j\right]v^i - L\right]_{\boldsymbol{v}=\boldsymbol{v}(\boldsymbol{q},\boldsymbol{p})} \\
&= -\left[\frac{m}{f}\left[g_{0i} + g_{ij}v^j\right]v^i - L\right]_{\boldsymbol{v}=\boldsymbol{v}(\boldsymbol{q},\boldsymbol{p})} \\
&= -\left[\frac{m}{f}\left[g_{0i}v^i + g_{ij}v^i v^j - g_{00} - 2g_{0i}v^i - g_{ik}v^i v^k\right]\right]_{\boldsymbol{v}=\boldsymbol{v}(\boldsymbol{q},\boldsymbol{p})} \\
&= -\left[\frac{m}{f}\left[-g_{0i}v^i + g_{ij}v^i v^j - g_{00}\right]\right]_{\boldsymbol{v}=\boldsymbol{v}(\boldsymbol{q},\boldsymbol{p})} \\
&= \left[\frac{m}{f}\left[g_{00} + g_{0i}v^i\right]\right]_{\boldsymbol{v}=\boldsymbol{v}(\boldsymbol{q},\boldsymbol{p})}
\end{aligned}
$$

And that is the result from the problem set.

(b) Differential equation for W

$$
\begin{aligned}
m^2 &\stackrel{!}{=} \langle \partial W, \partial W \rangle \\
&= g^{00}[\partial_0 W][\partial_0 W] + 2g^{0i}[\partial_0 W][\partial_i W] + 2g^{ij}[\partial_i W][\partial_j W] \\
&= g^{00}H^2 - 2g^{0i}Hp_i + g^{ij}p_i p_j
\end{aligned}
$$

I will now insert all the H and $\boldsymbol{p}$.

$$
= \frac{m^2}{f^2}\left\{g^{00}\left[g_{00} + g_{0i}v^i\right]^2 + 2g^{0j}\left[g_{00} + g_{0i}v_i\right]\left[g_{0j} + g_{ij}v^i\right] + g^{ij}\left[g_{0i} + g_{ki}v^k\right]\left[g_{0j} + g_{kj}v^k\right]\right\}
$$

The terms in the big group have to equal f^2 so that the result really becomes m^2. I will expand everything and see what cancels.

$$= \frac{m^2}{f^2}\left\{g^{00}\left[[g_{00}]^2 + 2g_{00}g_{0i}v^i + \left[g_{0i}v^i\right]^2\right] + 2g^{0j}\left[g_{00}g_{0j} + g_{00}g_{ij}v^i + g_{0j}g_{0i}v^i + g_{0i}v^i g_{kj}v^k\right]\right.$$
$$\left. + g^{ij}\left[g_{0i}g_{0j} + g_{0i}g_{kj}v^k + g_{0j}g_{ki}v^k + g_{ki}v^k g_{lj}v^l\right]\right\}$$

Indices can now be raised, and some combinations of $\boldsymbol{g}$ become $\boldsymbol{\delta}$.

$$= \frac{m^2}{f^2}\left[g_{00} + 2g_{0i}v^i + g^{00}\left[g_{0i}v^i\right]^2 + 2g_{00} + g_{00}v^0 + 2g_{0i}v^i + 2g_{0i}v^i v^0 + g_{00} + g_{k0}v^k + g_{k0}v^k + g_{kl}v^k v^l\right]$$

Using that $v^0 = 1$ and renaming the dummy indices, a lot of this can be written more compact:

$$= \frac{m^2}{f^2}\left[5g_{00} + 8g_{0i}v^i + g^{00}\left[g_{0i}v^i\right]^2 + g_{kl}v^k v^l\right]$$

That is getting kind of into the desired direction, but not close enough.

(c) Differential equation for $S(r)$

$\boldsymbol{g}$ is now diagonal. That makes the previous differential equation much easier.

$$\begin{aligned} m^2 &= \langle \partial W, \partial W \rangle \\ &= g^{00}\left[\frac{\partial W}{\partial t}\right]^2 + g^{11}\left[\frac{\partial W}{\partial r}\right]^2 + g^{22}\left[\frac{\partial W}{\partial \theta}\right]^2 + g^{33}\left[\frac{\partial W}{\partial \phi}\right]^2 \end{aligned}$$

The separation ansatz

$$W(t,r,\theta,\phi) = -Et + S(r) + l\arccos\big(y(\theta,\phi)\big)$$

with

$$y(\theta,\phi) = \cos(\alpha)\cos(\theta) + \sin(\alpha)\sin(\theta)\sin(\phi)$$

and α being a constant will give me chain rule contributions for the last two summands. I factor those out.

$$= g^{00}E^2 + g^{11}\left[\frac{\partial S}{\partial r}\right]^2 + \left[\frac{\partial W}{\partial y}\right]^2 \left[g^{22}\left[\frac{\partial y}{\partial \theta}\right]^2 + g^{33}\left[\frac{\partial y}{\partial \phi}\right]^2\right]$$

The derivative of arccos is given as:

$$\arccos'(x) = -\frac{1}{\sqrt{1-x^2}}.$$

With that:

$$m^2 = g^{00}E^2 + g^{11}\left[\frac{\partial S}{\partial r}\right]^2 + \frac{l}{1-y^2}\left[g^{22}\left[\frac{\partial y}{\partial \theta}\right]^2 + g^{33}\left[\frac{\partial y}{\partial \phi}\right]^2\right]$$

$$m^2 = \frac{1}{1-\frac{r_S}{r}}E^2 - \left[1-\frac{r_S}{r}\right]\left[\frac{\partial S}{\partial r}\right]^2 + \frac{1}{r^2}\frac{l}{1-y^2}\left[\left[\frac{\partial y}{\partial \theta}\right]^2 + \csc(\theta)^2\left[\frac{\partial y}{\partial \phi}\right]^2\right]$$

$$m^2 r^2 = \frac{r^2}{1-\frac{r_S}{r}}E^2 - r^2\left[1-\frac{r_S}{r}\right]\left[\frac{\partial S}{\partial r}\right]^2 - \frac{l}{1-y^2}\left[\left[\frac{\partial y}{\partial \theta}\right]^2 + \csc(\theta)^2\left[\frac{\partial y}{\partial \phi}\right]^2\right]$$

Now I separate the radial and angular parts.

$$\frac{r^2}{1-\frac{r_S}{r}}E^2 - r^2\left[1-\frac{r_S}{r}\right]\left[\frac{\partial S}{\partial r}\right]^2 - m^2r^2 = \frac{l}{1-y^2}\left[\left[\frac{\partial y}{\partial \theta}\right]^2 + \csc(\theta)^2\left[\frac{\partial y}{\partial \phi}\right]^2\right]$$

Since both sides depend on different variables, they both have to equal a constant, that I will name c_1.

$$\frac{r^2}{1-\frac{r_S}{r}}E^2 - r^2\left[1-\frac{r_S}{r}\right]\left[\frac{\partial S}{\partial r}\right]^2 - m^2 r^2 = c_1$$

Now I can isolate S:

$$\left[\frac{\partial S}{\partial r}\right]^2 = \frac{E^2}{\left[1-\frac{r_S}{r}\right]^2} - \frac{c_1}{r^2\left[1-\frac{r_S}{r}\right]} - \frac{m^2}{1-\frac{r_S}{r}}$$

Taking the square root, integrating ...

$$S(r') = \int_0^r \mathrm{d}r' \sqrt{\frac{E^2}{\left[1-\frac{r_S}{r}\right]^2} - \frac{c_1}{r^2\left[1-\frac{r_S}{r}\right]} - \frac{m^2}{1-\frac{r_S}{r}}}$$

So the whole W now looks like this:

$$W = -Et + \int_0^r \mathrm{d}r' \sqrt{\frac{E^2}{\left[1-\frac{r_S}{r}\right]^2} - \frac{c_1}{r^2\left[1-\frac{r_S}{r}\right]} - \frac{m^2}{1-\frac{r_S}{r}}} - l \arccos\big(y(\theta, \phi)\big).$$

(d) Trajectories

I am supposed to calculate P_2. I do not even know which of my parameters is Q_2. So my parameters are E, c_1 and I think that the l is yet another one of those parameters. But it could also be α. One of those parameters is meaningless, and I should only get 3. That is what I read in (Kuypers 2010).

So I will just start with l. That might be Q_3 or so.

$$\begin{aligned}
\frac{\partial W}{\partial l} &= \arccos(y) = -P_3 \\
y &= \cos(-P_3) \\
y &= \cos(P_3) \\
\cos(\alpha)\cos(\theta) + \sin(\alpha)\sin(\theta)\sin(\phi) &= \cos(P_3)
\end{aligned}$$

I can now separate θ and ϕ to either side of the equation.

$$\sin(\phi) = \cos(P_3)\sin(\alpha)\sin(\theta) - \cot(\alpha)\cot(\theta)$$

That gives me a relationship between θ and ϕ, but nothing with respect to r.

Then I tried to use c_1 as the Q_2. The problem is that c_1 does not occur in the angular term. So this will give me only a relationship from r with t at best.

$$\frac{\partial W}{\partial c_1} = -P_2$$
$$-\int \mathrm{d}r \frac{1}{\sqrt{\frac{E^2}{\left[1-\frac{r_S}{r}\right]^2} - \frac{c_1}{r^2\left[1-\frac{r_S}{r}\right]} - \frac{m^2}{1-\frac{r_S}{r}}}} \frac{1}{r^2\left[1-\frac{r_S}{r}\right]} = -P_2$$

Even if I could solve this with respect to r, there would not be any angular part.

References

Kuypers, Friedhelm (2010). *Klassische Mechanik*. 9th ed. Wileyy-VCH Verlag GmbH & Co. KGaA. ISBN: 978-3-527-40989-1.