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This problem set is not reviewed by a tutor. This is just what I have turned in.

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[disclaimer]

physics754 – Problem Set 4

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1 Diffeomorphisms and vector fields

The various uses of y are not defined in the problem set, so I will do that now to avoid confusion:

- $X : \mathbb{R}^4 \mapsto \mathbb{R}^4$ is a vector field.
- $y : \mathbb{R} \mapsto \mathbb{R}^4$ is a trajectory.
- $f_\tau : \mathbb{R}^4 \mapsto \mathbb{R}^4$ is a diffeomorphism $\forall \tau \in \mathbb{R}$.
- $f : \mathbb{R}^4 \times \mathbb{R} \mapsto \mathbb{R}^4$ is a $\mathbb{R}$-parameter family of diffeomorphisms.
- $\tilde{y} : \mathbb{R} \times \mathbb{R}^4 \mapsto \mathbb{R}^4$ is a $\mathbb{R}^4$-parameter family of trajectories starting at x. This is used by f to create a $\mathbb{R}$-parameter family of diffeomorphisms.
- $\hat{y} \in \mathbb{R}^4$ is a vector in the image of $\tilde{y}$.
- $x \in \mathbb{R}^4$ is a vector in the domain of $\tilde{y}$, which is the same space as the image, since $y(0) = x$.
- $\tau \in \mathbb{R}$ is a scalar and parameter for y.

The ambiguity of round parentheses arising because they are used for both scoping and function application makes the equation a bit hard to read. I will use a Mathematica inspired notation that does not contradict the usual notation: Round parentheses are used only for function application, square brackets are used for scoping and grouping.

I can now safely omit using different fonts for scalars, vectors and tensors of higher rank since every symbol is defined.

1.1

With all that set, the first problem nearly does itself:

$$\begin{aligned} X(\hat{y}) &= \left[\frac{\mathrm{d}}{\mathrm{d}\tau} f_\tau\right]\left(f_\tau^{-1}(\hat{y})\right) \\ &= \left[\frac{\mathrm{d}}{\mathrm{d}\tau} \tilde{y}\right]\left(\tilde{y}^{-1}(\hat{y})\right) \end{aligned}$$

y is a solution of (12) from the problem set. $\tilde{y}$ additionally solves (13) as well. So $\tilde{y}$ will solve (12). So

$$\left[\frac{\mathrm{d}}{\mathrm{d}\tau} \tilde{y}\right](x,\tau) = [X \circ \tilde{y}](x,\tau)$$

and therefore

$$\begin{aligned} &= [X \circ \tilde{y}]\left(\tilde{y}^{-1}(\hat{y})\right) \\ &= [X \circ \tilde{y} \circ \tilde{y}^{-1}](\hat{y}) \\ &= X(\hat{y}) \end{aligned}$$

$f_0(x)$ is just $\tilde{y}(x,0)$. Since $y(0) = x$, $\tilde{y}(x,0)$ reduces to x. Therefore, f_0 is indeed the identity transformation. That shows $f_0 = \mathbb{1}$.

1.2

Show that for any g:

$$\left.\frac{\mathrm{d}}{\mathrm{d}\tau} f_\tau^* g_{\mu\nu}\right|_{\tau=0} = 2X_{(\mu;\nu)},$$

where I have used the symmetrization parentheses.

The right hand side is given by:

$$2X_{(\mu;\nu)} = 2X_{(\mu,\nu)} - 2\Gamma^\lambda_{(\mu\nu)} X_\lambda$$

Now I will turn to the left hand side, where I will evaluate it at x.

$$\mathrm{RHS} = \left.\frac{\mathrm{d}}{\mathrm{d}\tau} f_\tau^* g_{\mu\nu}\right|_{\tau=0} (x)$$

Now I will apply the transformation to g.

$$= \frac{\mathrm{d}}{\mathrm{d}\tau} g_{\alpha\beta}\left(f_\tau(x)\right) f_{\tau}{}^{\alpha}{}_{,\mu}(x)\, f_{\tau}{}^{\beta}{}_{,\nu}(x)\bigg|_{\tau=0}$$

The lower index τ is a little messy, so I will expand f in terms of $\tilde{y}$, introducing even more explicit parameters.

$$= \frac{\mathrm{d}}{\mathrm{d}\tau} g_{\alpha\beta}\left(\tilde{y}(x,\tau)\right) \tilde{y}^{\alpha}_{,\mu}(x,\tau)\, \tilde{y}^{\beta}_{,\nu}(x,\tau)\bigg|_{\tau=0}$$

Total differentiation will act on all three factors, since all of them depend on τ. This gives three terms:

$$= \left[\frac{\mathrm{d}}{\mathrm{d}\tau} g_{\alpha\beta}\left(\tilde{y}(x,\tau)\right)\right] \tilde{y}^{\alpha}_{,\mu}(x,\tau)\, \tilde{y}^{\beta}_{,\nu}(x,\tau) + g_{\alpha\beta}\left(\tilde{y}(x,\tau)\right) \left[\frac{\mathrm{d}}{\mathrm{d}\tau} \tilde{y}^{\alpha}_{,\mu}(x,\tau)\right] \tilde{y}^{\beta}_{,\nu}(x,\tau)$$
$$+ g_{\alpha\beta}\left(\tilde{y}(x,\tau)\right) \tilde{y}^{\alpha}_{,\mu}(x,\tau) \left[\frac{\mathrm{d}}{\mathrm{d}\tau} \tilde{y}^{\beta}_{,\nu}(x,\tau)\right]\Bigg|_{\tau=0}$$

To simplify, I will now apply the $\tau = 0$ to all the terms that make it possible at this stage. $\tilde{y}(x,0)$ is just $y(0)$, which was defined to be x. This simplification will be in the $g(\ldots)$. The other thing I can do right now is to look at $\tilde{y}^{\alpha}_{,\mu}(x,\tau)$: When $\tau = 0$, $\tilde{y}$ reduces to x, so that will be a δ^{α}_{μ}.

$$= \left[\frac{\mathrm{d}}{\mathrm{d}\tau} g_{\alpha\beta}\left(\tilde{y}(x,\tau)\right)\right]\Bigg|_{\tau=0} \delta^{\alpha}_{\mu}\, \delta^{\beta}_{\nu} + g_{\alpha\beta}(x) \left[\frac{\mathrm{d}}{\mathrm{d}\tau} \tilde{y}^{\alpha}_{,\mu}(x,\tau)\right]\Bigg|_{\tau=0} \delta^{\beta}_{\nu}$$
$$+ g_{\alpha\beta}(x)\, \delta^{\alpha}_{\mu} \left[\frac{\mathrm{d}}{\mathrm{d}\tau} \tilde{y}^{\beta}_{,\nu}(x,\tau)\right]\Bigg|_{\tau=0}$$

The τ-differentiation for $\tilde{y}$ are given via the differential equation that defines y.

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\,\tilde{y}^{\alpha}_{,\mu}(x,\tau) = \frac{\mathrm{d}}{\mathrm{d}\tau}\,\frac{\partial}{\partial x^{\mu}}\,\tilde{y}^{\alpha}(x,\tau).$$

The derivatives should commute, so this becomes

$$\frac{\partial}{\partial x^{\mu}}\,\frac{\mathrm{d}}{\mathrm{d}\tau}\,\tilde{y}^{\alpha}(x,\tau) = \frac{\partial}{\partial x^{\mu}}\,X^{\alpha}\big(y(\tau)\big)$$

via the differential equation (12). There is no problem with setting $\tau = 0$ here. $y(\tau)$ becomes x. I will write the result as $X^{\alpha}_{,\mu}(x)$.

$$\begin{aligned}
&= \frac{\mathrm{d}}{\mathrm{d}\tau}\,g_{\mu\nu}\big(\tilde{y}(x,\tau)\big)\bigg|_{\tau=0} + g_{\alpha\nu}(x)\,X^{\alpha}_{,\mu}(x) + g_{\mu\beta}(x)\,X^{\beta}_{,\nu}(x) \\
&= \frac{\mathrm{d}}{\mathrm{d}\tau}\,g_{\mu\nu}\big(\tilde{y}(x,\tau)\big)\bigg|_{\tau=0} + 2X_{(\mu,\nu)}(x)
\end{aligned}$$

Now comes the part where the Γ have to appear to make it fit. Let $\hat{y} := \tilde{y}(x,\tau)$ from this point. That lets me write the chain rule:

$$= \frac{\partial g_{\mu\nu}}{\partial \hat{y}^{\lambda}}\,\frac{\partial \hat{y}^{\lambda}}{\partial \tau}\bigg|_{\tau=0}(x) + 2X_{(\mu,\nu)}(x)$$

The τ-derivative of $\hat{y}$ can be expressed by the differential equation. g will be differentiated with respect to x^{λ} in the $\tau = 0$ case.

$$\begin{aligned}
&= g_{\mu\nu,\lambda}(x)\,X^{\lambda}(x) + 2X_{(\mu,\nu)}(x) \\
&= \Big[g_{\mu\nu,\lambda}\,X^{\lambda} + 2X_{(\mu,\nu)}\Big](x)
\end{aligned}$$

The Γ terms are missing, since there is only derivative of g.

1.3 Action

Given

$$S_{\mathrm{gr}}(g) := -\frac{1}{16\pi G}\int\limits_{\mathbb{R}^4} \mathrm{d}^4x\,\sqrt{|g|}\mathscr{R}(g)$$

I have to show that

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\,S_{\mathrm{gr}}(f^{*}_{\tau}g)\bigg|_{\tau=0} = \frac{2}{16\pi G}\int\limits_{\mathbb{R}^4} \mathrm{d}^4x\,\sqrt{|g|}G^{\mu\nu}X_{(\mu;\nu)}.$$

The various forks induced by the product rule will make this calculation nonlinear, I will try to perform an in-order traversal of all the steps while dragging intermediate results along. First I will put the transformed g into the definition of the action.

$$\frac{\mathrm{d}}{\mathrm{d}\tau} S_{\mathrm{gr}}(f_\tau^* g)\bigg|_{\tau=0} = -\frac{1}{16\pi G} \frac{\mathrm{d}}{\mathrm{d}\tau} \int_{\mathbb{R}^4} \mathrm{d}^4 x \, \sqrt{|f_\tau^* g|} \mathscr{R}(f_\tau^* g)\bigg|_{\tau=0}$$

Given that the result has a similar prefactor and an integration, I will focus my attention to the integrand. If some summand will be left that does not fit into the result, I will see whether the integral over it does vanish. First I will assume that I can interchange the limit, the differentiation and the integration.

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}\tau} \sqrt{|f_\tau^* g|} \mathscr{R}(f_\tau^* g)\bigg|_{\tau=0} \\ &= \left[\frac{\mathrm{d}}{\mathrm{d}\tau} \sqrt{|f_\tau^* g|}\right] \mathscr{R}(f_\tau^* g)\bigg|_{\tau=0} + \sqrt{|f_\tau^* g|} \frac{\mathrm{d}}{\mathrm{d}\tau} \mathscr{R}(f_\tau^* g)\bigg|_{\tau=0} \end{aligned}$$

The terms that are not differentiated to not have to wait until τ is set to zero. Once this is done f will become the identity transformation, leaving the g untouched.

$$= \left[\frac{\mathrm{d}}{\mathrm{d}\tau} \sqrt{|f_\tau^* g|}\right]_{\tau=0} \mathscr{R}(g) + \sqrt{|g|} \left[\frac{\mathrm{d}}{\mathrm{d}\tau} \mathscr{R}(f_\tau^* g)\right]_{\tau=0}$$

Using Jacobi's formula for the derivative of determinants that we proved on an earlier exercise set I will work on the first term in square brackets.

$$= \left[\frac{1}{\sqrt{|f_\tau^* g|}} \operatorname{Tr}\left([f_\tau^* g]^\dagger \frac{\mathrm{d}}{\mathrm{d}\tau} f_\tau^* g\right)\right]_{\tau=0} \mathscr{R}(g) + \sqrt{|g|} \left[\frac{\mathrm{d}}{\mathrm{d}\tau} \mathscr{R}(f_\tau^* g)\right]_{\tau=0}$$

The first part can be evaluated at $\tau = 0$ now. The last derivative was derived in the prior parts, so that I just plug that in.

$$= \frac{1}{\sqrt{|g|}} \operatorname{Tr}\left(g^\dagger 2 X_{(\mu;\nu)}\right) \mathscr{R}(g) + \sqrt{|g|} \left[\frac{\mathrm{d}}{\mathrm{d}\tau} \mathscr{R}(f_\tau^* g)\right]_{\tau=0}$$

That trace will totally contract the g and the X, giving a scalar:

$$= \frac{2}{\sqrt{|g|}} g^{\mu\nu} \mathscr{R}(g) X_{(\mu;\nu)} + \sqrt{|g|} \left[\frac{\mathrm{d}}{\mathrm{d}\tau} \mathscr{R}(f_\tau^* g)\right]_{\tau=0}$$

I swapped the g and the X in the first summand, since I want to factor out X later on since $g\mathscr{R}$ appear in the Einstein tensor which gets contracted with X in the final expression. The next work will be done on the second summand. First, I expand the Ricci scalar in terms of the Ricci tensor.

$$= \frac{2}{\sqrt{|g|}} g^{\mu\nu} \mathscr{R}(g) X_{(\mu;\nu)} + \sqrt{|g|} \left[\frac{\mathrm{d}}{\mathrm{d}\tau} \left[f_\tau^* g^{\mu\nu} \right] R_{\mu\nu}(f_\tau^* g) \right]_{\tau=0}$$

While I apply product rule, I will evaluate at $\tau = 0$ in this step as well. The ellipsis stands for the first summand of the above expression.

$$= \ldots + \sqrt{|g|} \left[\frac{\mathrm{d}}{\mathrm{d}\tau} f_\tau^* g^{\mu\nu} \right]_{\tau=0} R_{\mu\nu}(g) + \sqrt{|g|} g^{\mu\nu} \left[\frac{\mathrm{d}}{\mathrm{d}\tau} R_{\mu\nu}(f_\tau^* g) \right]_{\tau=0}$$

The dual metric tensor is defined as $g^{\alpha\beta} g_{\beta\gamma} = \delta^\alpha_\gamma$. This also holds for the transformed metric tensor. *The δ is not transformed since it is a symbol and not a tensor in general? Here, I want the two terms to be the inverse map of each other, so that is a special case, right?* Transforming that identity yields

$$\left[f_\tau^* g^{\alpha\beta} \right] \left[f_\tau^* g_{\beta\gamma} \right] = \delta^\alpha_\gamma.$$

The derivative with respect to τ gives, applying the product rule right away and evaluating at $\tau = 0$:

$$\left[\frac{\mathrm{d}}{\mathrm{d}\tau} f_\tau^* g^{\alpha\beta} \right]_{\tau=0} g_\beta\gamma + g^{\alpha\beta} \left[\frac{\mathrm{d}}{\mathrm{d}\tau} f_\tau^* g_{\beta\gamma} \right]_{\tau=0} = 0.$$

Using the known expression for the derivative in the second summand, this can be written as

$$\left[\frac{\mathrm{d}}{\mathrm{d}\tau} f_\tau^* g^{\alpha\eta} \right]_{\tau=0} = -2X^{(\alpha;\eta)}.$$

The $^{;\eta}$ includes the metric tensor. Now this goes back into the dragged along expression. Since the line is long enough, I included the omitted part again. I also raised and lowered the indices on R and X respectively, to remove the $^{;\eta}$ that was not featured in the lecture so far.

$$= \frac{2}{\sqrt{|g|}} g^{\mu\nu} \mathscr{R}(g) X_{(\mu;\nu)} - 2\sqrt{|g|} R^{\mu\nu}(g) X_{(\mu;\nu)} + \sqrt{|g|} g^{\mu\nu} \left[\frac{\mathrm{d}}{\mathrm{d}\tau} R_{\mu\nu}(f_\tau^* g) \right]_{\tau=0}$$

The middle summand is the part with R within G. It has the correct sign and prefactor, note the sign change from the definition to the result. Now the first and last term have to turn out to be $\sqrt{|g|}\mathscr{R} g X$. The factor $\frac{1}{2}$ that is within G will be supplied by the symmetrization of X. Omitting the first two summands …

$$= \ldots + \sqrt{|g|} g^{\mu\nu} \left[\frac{\mathrm{d}}{\mathrm{d}\tau} R_{\mu\nu}(f_\tau^* g) \right]_{\tau=0}$$

Let $\tilde{g} := f_\tau^* g$. With that, I attempt to use chain rule for the last term. R depends on all 16 components of g. The chain rule would have to reflect that. With the following, I indeed sum over 16 entries, which seems like the right direction.

$$= \ldots + \sqrt{|g|} g^{\mu\nu} \left[\frac{\partial R_{\mu\nu}}{\partial \tilde{g}_{\alpha\beta}} \frac{\mathrm{d}\tilde{g}_{\alpha\beta}}{\mathrm{d}\tau} \right]_{\tau=0}$$

In the last partial derivative, the definition of $\tilde{g}$ can be expanded again:

$$= \ldots + \sqrt{|g|} g^{\mu\nu} \left[\frac{\partial R_{\mu\nu}}{\partial \tilde{g}_{\alpha\beta}} \frac{\mathrm{d}}{\mathrm{d}\tau} f_\tau^* g_{\alpha\beta} \right]_{\tau=0}$$

That derivative was used often now, and is given by $2X_{(\alpha;\beta)}$.

$$= \ldots + 2\sqrt{|g|} g^{\mu\nu} \left[\frac{\partial R_{\mu\nu}}{\partial \tilde{g}_{\alpha\beta}} \right]_{\tau=0} X_{(\alpha;\beta)}$$

I do not see anything holding me back from evaluating that at $\tau = 0$ now and getting plain g back.

$$= \frac{2}{\sqrt{|g|}} g^{\mu\nu} \mathscr{R}(g) X_{(\mu;\nu)} - 2\sqrt{|g|} R^{\mu\nu}(g) X_{(\mu;\nu)} + 2\sqrt{|g|} g^{\mu\nu} \frac{\partial R_{\mu\nu}}{\partial g_{\alpha\beta}} X_{(\alpha;\beta)}$$

This is the part where I finally got stuck. I have tried to expand R in terms of Γ and to expand those in terms of g again. The derivatives of g with respect to g only give me δ-symbols. Since within the Γ, there are derivatives of g, I tried to interchange the derivatives. But $\partial_m u\delta = 0$, so the terms vanished.

There were also terms where I had to differentiate the dual metric with respect to the metric, and I have no idea how I could gain the elements of that (0, 4)-tensor.

I assume that what I have tried is either a dead end or relatively close to the correct way of doing it, I will stop here and wait to see this in the exercise group.

1.4

In exercise H.6, part (b), I have shown that $G^{\mu\nu}_{;\mu}$ vanishes. The Einstein tensor is symmetric, since both g and the Ricci tensor R are symmetric. Therefore $G^{\mu\nu}_{;\mu} = G^{\mu\nu}_{;\nu}$ if you use the symmetry and relabel the indices. Since $G \propto T$, the vanishing divergence must apply for T as well.

This seems a little unrelated to the previous problems, so I think that there is another way to show this using the relations given. The action should be invariant under any diffeomorphism. Therefore, the derivative of the action with respect to the parameter τ should vanish.