Disclaimer

This is a problem set (as turned in) for the module physics754.

This problem set is not reviewed by a tutor. This is just what I have turned in.

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[disclaimer]

physics754 – Problem Set 3

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1 Bianchi Identities

1.1 Show the differential Bianchi identity

I have to show that:

$$R_{\alpha\beta\mu\nu;\lambda} + R_{\alpha\beta\lambda\mu;\nu} + R_{\alpha\beta\nu\lambda;\mu} = 0.$$

Since this tensor is antisymmetric in its last two indices, I think it is the same as $R_{\alpha\beta[\mu\nu;\lambda]} = 0$. The hint says that I only have to show

$$R_{\alpha\beta\mu\nu,\lambda} + R_{\alpha\beta\lambda\mu,\nu} + R_{\alpha\beta\nu\lambda,\mu} = 0.$$

Since $\boldsymbol{\Gamma}(\boldsymbol{x}) = \boldsymbol{0}$ and only $\partial\boldsymbol{\Gamma}(\boldsymbol{x}) \neq \boldsymbol{0}$, this individual summands reduce to:

$$R^\alpha_{\beta\mu\nu,\lambda} = \Gamma^\alpha_{\beta\nu,\mu,\lambda} - \Gamma^\alpha_{\beta\mu,\nu,\lambda}, \quad R^\alpha_{\beta\lambda\mu,\nu} = \Gamma^\alpha_{\beta\mu,\lambda,\nu} - \Gamma^\alpha_{\beta\lambda,\mu,\nu}, \quad R^\alpha_{\beta\nu\lambda,\mu} = \Gamma^\alpha_{\beta\lambda,\nu,\mu} - \Gamma^\alpha_{\beta\nu,\lambda,\mu}.$$

Now I just add them all together and get

$$+\Gamma^\alpha_{\beta\nu,\mu,\lambda} - \Gamma^\alpha_{\beta\mu,\nu,\lambda} + \Gamma^\alpha_{\beta\mu,\lambda,\nu} - \Gamma^\alpha_{\beta\lambda,\mu,\nu} + \Gamma^\alpha_{\beta\lambda,\nu,\mu} - \Gamma^\alpha_{\beta\nu,\lambda,\mu},$$

which can be rearranged (last summand into second place) to

$$+\Gamma^\alpha_{\beta\nu,\mu,\lambda} - \Gamma^\alpha_{\beta\nu,\lambda,\mu} - \Gamma^\alpha_{\beta\mu,\nu,\lambda} + \Gamma^\alpha_{\beta\mu,\lambda,\nu} - \Gamma^\alpha_{\beta\lambda,\mu,\nu} + \Gamma^\alpha_{\beta\lambda,\nu,\mu}$$

which is 0 given the commutative property of the partial derivatives.

1.2 Argument for equation (7)

It should be shown that $G^{\mu\nu}_{;\mu} = 0$ holds. Given the definition of $\boldsymbol{G}$, this is:

$$\nabla_\mu G^{\mu\nu} = \nabla_\mu g^{\mu\alpha} g^{\nu\beta} R_{\alpha\beta} - \frac{1}{2}\nabla_\mu g^{\mu\nu}\mathscr{R}$$

All the $\nabla \boldsymbol{g}$ terms are 0.

$$
\begin{aligned}
&= g^{\mu\alpha} g^{\nu\beta} \nabla_\mu R_{\alpha\beta} - \frac{1}{2} g^{\mu\nu} \nabla_\mu \mathscr{R} \\
&= g^{\mu\alpha} g^{\nu\beta} \nabla_\mu R^{\lambda}_{\alpha\lambda\beta} + \nabla_\mu g^{\beta\nu} R_{\beta\nu} \\
&= g^{\mu\alpha} g^{\nu\beta} \nabla_\mu R^{\lambda}_{\alpha\lambda\beta} + g^{\beta\nu} \nabla_\mu R_{\beta\nu} \\
&= g^{\mu\alpha} g^{\nu\beta} \nabla_\mu R^{\lambda}_{\alpha\lambda\beta} + g^{\beta\nu} \nabla_\mu R^{\lambda}_{\beta\lambda\nu}
\end{aligned}
$$

I found this easier without all the explicit occurrences of $\boldsymbol{g}$. That and $\boldsymbol{\nabla}$ commute, so I can make this implicit by raising and lowering the indices, keeping track of their order.

$$
\begin{aligned}
&= \nabla_\mu R^{\lambda\mu}{}_{\lambda}{}^{\nu} - \frac{1}{2} \nabla^\nu R^{\lambda\rho}{}_{\lambda\rho} \\
&= R^{\lambda\mu}{}_{\lambda}{}^{\nu}{}_{;\mu} + \frac{1}{2}(R^{\lambda\rho\nu}{}_{\lambda;\rho} + R^{\lambda\rho}{}_{\rho}{}^{\nu}{}_{;\lambda}) \\
&= R^{\lambda\mu}{}_{\lambda}{}^{\nu}{}_{;\mu} + \frac{1}{2}(R^{\lambda\rho\nu}{}_{\lambda;\rho} + R^{\rho\lambda}{}_{\lambda}{}^{\nu}{}_{;\rho}) \\
&= R^{\lambda\mu}{}_{\lambda}{}^{\nu}{}_{;\mu} + \frac{1}{2}(R^{\lambda\rho\nu}{}_{\lambda;\rho} + R^{\lambda\rho\nu}{}_{\lambda;\rho}) \\
&= R^{\lambda\mu}{}_{\lambda}{}^{\nu}{}_{;\mu} + R^{\lambda\rho\nu}{}_{\lambda;\rho} \\
&= R^{\lambda\rho}{}_{\lambda}{}^{\nu}{}_{;\rho} - R^{\lambda\rho}{}_{\lambda}{}^{\nu}{}_{;\rho} \\
&= 0
\end{aligned}
$$

2 Normal Coordinates

2.1 Conclusion

If all the geodesics are straight lines, that means that the covariant and partial derivative are equal. Since

$$\nabla_\mu \boldsymbol{e}_i = \partial_\mu \boldsymbol{e}_i + \Gamma^k_{\mu i} \boldsymbol{e}_k,$$

the Christoffel-symbols have to be zero in that case. If that is the case, the relation holds.

I can construct the symbol from equation (16) like so:

$$(\nabla_\alpha y^\beta) y^\alpha = (\partial_\alpha y^\beta) y^\alpha + \Gamma^\beta_{\alpha\lambda} y^\alpha y^\nabla.$$

If I require that $\nabla \sim \partial$ holds in this case, the term with Γ has to be zero. I can require that in this coordinate system, since geodesics are straight lines. I think that also means that we are in a sort of "inertial reference frame", where the covariant derivatives are just the partial derivatives.

2.2 Implication

The Christoffel-symbol is given as

$$\Gamma^{\gamma}_{\alpha\beta} = \frac{1}{2} g^{\gamma\delta} \left(g_{\beta\delta,\alpha} + g_{\alpha\delta,\beta} + g_{\alpha\beta,\delta} \right).$$

The property that has to be shown is that there is no “slope” in $\boldsymbol{g}$ at the origin. I could not come up with a sound reason why this has to be the case if the previous relation holds.