Disclaimer

This is a problem set (as turned in) for the module physics754.

This problem set is not reviewed by a tutor. This is just what I have turned in.

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[disclaimer]

physics754 – Problem Set 2

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2014-04-28
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When writing down tensor components, I have seen $R^{\delta}_{\alpha\beta\gamma}$ as well as $R_{\alpha\beta\gamma}{}^{\delta}$ and maybe $R^{\delta}{}_{\alpha\beta\gamma}$. Does that make a difference? If you raise and lower indices and want to keep their position fixed, this "splitting up" the lower and upper indices seems to clarify. But do we need this really?

1 Transformation of the Christoffel-Symbols

1.1 Compute $\Gamma^{\alpha}_{\beta\gamma}(\varphi^* g)$

I assume that this means that I should compose Γ with the transformed g.

The transformed g is given by:

$$(\varphi^* g)_{\mu\nu}(x) = g_{\alpha\beta}(\varphi(x)) \frac{\partial \varphi^\alpha}{\partial x^\mu} \frac{\partial \varphi^\beta}{\partial x^\nu} = g_{\alpha\beta}(\varphi(x)) \varphi^{\alpha}_{,\mu}(x) \varphi^{\beta}_{,\nu}(x).$$

The transformed dual[1] g is similarly obtained:

$$(\varphi^* g)^{\mu\nu} = g^{\mu\nu}(\varphi(x)) \frac{\partial x^\gamma}{\partial \varphi^\mu} \frac{\partial x^\delta}{\partial \varphi^\nu}.$$

With that, I construct Γ using its definition, except that I use the transformed g wherever g is used in the definition.

$$\Gamma^{\gamma}_{\beta\alpha}(\varphi^* g) = \frac{1}{2} g^{\epsilon\eta}(\varphi(x)) \frac{\partial x^\gamma}{\partial \varphi^\mu} \frac{\partial x^\delta}{\partial \varphi^\nu} \Big(\partial_\beta g_{\mu\nu}(\varphi(x)) \varphi^{\mu}_{,\delta}(x) \varphi^{\nu}_{,\alpha}(x)$$
$$+ \partial_\alpha g_{\mu\nu}(\varphi(x)) \varphi^{\mu}_{,\delta}(x) \varphi^{\nu}_{,\beta}(x) + \partial_\gamma g_{\mu\nu}(\varphi(x)) \varphi^{\mu}_{,\alpha}(x) \varphi^{\nu}_{,\beta}(x) \Big)$$

In a case like

$$\partial_\beta g_{\mu\nu}(\varphi(x)) \varphi^{\mu}_{,\delta}(x) \varphi^{\nu}_{,\alpha}(x),$$

[1] Is "dual" correct at this point?

where I have expanded g in front of a partial derivative, do I have to use the product rule next? But that would involve second derivatives of φ, which seems arcane to me right now.

1.2 Do the Christoffel-symbols transform like tensor(field)s?

Transforming Γ like a tensor:

$$(\varphi^* \Gamma)^{\alpha}_{\beta\gamma}(x) = \Gamma^{\gamma}_{\epsilon\eta}(\varphi(x)) \frac{\partial \varphi^{\alpha}}{\partial x^{\delta}} \frac{\partial x^{\epsilon}}{\partial \varphi^{\beta}} \frac{\partial x^{\eta}}{\partial \varphi^{\gamma}} .$$

Expanding Γ:

$$= \frac{1}{2} g^{\delta\chi}(\varphi(x)) \Big(\partial_\epsilon g_{\chi\eta}(\varphi(x)) + \partial_\eta g_{\chi\epsilon}(\varphi(x)) - \partial_\chi g_{\epsilon\eta}(\varphi(x)) \Big) \frac{\partial \varphi^{\alpha}}{\partial x^{\delta}} \frac{\partial x^{\epsilon}}{\partial \varphi^{\beta}} \frac{\partial x^{\eta}}{\partial \varphi^{\gamma}} .$$

It looks different than Γ with the transformed metric. In that, there are four partial dervatives of φ, the transformed Γ only features three of them, though.

2 Curvature Tensor

I use the antisymmetrization notation that is used in (Penrose 2005) with a normalization of $1/n!$ so that it is idempotent, like so:

$$\nabla_{[\mu} \nabla_{\nu]} := \frac{1}{2} \Big(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu \Big).$$

2.1 Show that $2\nabla_{[\mu}\nabla_{\nu]} X^\rho = R^{\rho}_{\alpha\mu\nu} X^\alpha$

I will start with the $\mu\nu$ term and antisymmetrize that later.

$$\begin{aligned} \nabla_\nu X^\rho &= \partial_\nu X^\rho + X^\alpha \Gamma^{\rho}_{\alpha\nu} \\ \nabla_\mu \nabla_\nu X^\rho &= (\nabla_\mu \partial_\nu) X^\rho + \partial_\nu \nabla_\mu X^\rho + (\nabla_\mu X^\alpha) \Gamma^{\rho}_{\alpha\nu} + X^\alpha \nabla_\mu \Gamma^{\rho}_{\alpha\nu} \end{aligned}$$

Until here, I am not even sure whether I have do differentiate the ∂ again. Do I have to? Does ∇ and ∂ even commute, like I assumed? If I have to do the first and may not use the latter, how would I even write this down? Now I have to use the definition of ∇ again:

$$\begin{aligned} &= (\partial_\mu \partial_\nu + \partial_\alpha \Gamma^{\alpha}_{\nu\mu}) X^\rho + \partial_\nu (\partial_\mu X^\rho + x^\alpha \Gamma^{\rho}_{\alpha\mu}) + (\partial_\mu X^\alpha + X^\lambda \Gamma^{\alpha}_{\lambda\mu}) \Gamma^{\rho}_{\alpha\nu} \\ &\quad + X^\alpha (\partial_\mu \Gamma^{\rho}_{\alpha\nu} + \Gamma^{\lambda}_{\alpha\nu} \Gamma^{\rho}_{\lambda\nu} - \Gamma^{\rho}_{\lambda\nu} \Gamma^{\lambda}_{\alpha\mu} - \Gamma^{\rho}_{\alpha\lambda} \Gamma^{\lambda}_{\nu\mu}) \end{aligned}$$

Using the notation introduced earlier and using the symmetry of Γ in its lower indices as well as the commutative property of multiplication, I can can indicate terms that are already symmetric or antisymmetric in μ and ν. I hope that the partial derivatives commute, but I do not see a problem with that since they are orthogonal to each other.

$$\begin{aligned}&= (\partial_{(\mu}\partial_{\nu)} + \partial_\alpha \Gamma^\alpha_{(\mu\nu)})X^\rho + \partial_\nu(\partial_\mu X^\rho + x^\alpha \Gamma^\rho_{\alpha\mu}) + (\partial_\mu X^\alpha + X^\lambda \Gamma^\alpha_{\lambda\mu})\Gamma^\rho_{\alpha\nu} \\ &\quad + X^\alpha(\partial_\mu \Gamma^\rho_{\alpha\nu} - 2\Gamma^\lambda_{\alpha[\mu}\Gamma^\rho_{\nu]\lambda} - \Gamma^\rho_{\alpha\lambda}\Gamma^\lambda_{(\mu\nu)})\end{aligned}$$

At this point, the first big summand was created because I differentiated ∂ again. That term is totally symmetric, so that it will not show up in the antisymmetric part that is asked in the problem anyway. The last summand in the last big summand will drop out as well. These terms are the interesting ones:

$$= \partial_\nu(\partial_\mu X^\rho + x^\alpha \Gamma^\rho_{\alpha\mu}) + (\partial_\mu X^\alpha + X^\lambda \Gamma^\alpha_{\lambda\mu})\Gamma^\rho_{\alpha\nu} + X^\alpha(\partial_\mu \Gamma^\rho_{\alpha\nu} - 2\Gamma^\lambda_{\alpha[\mu}\Gamma^\rho_{\nu]\lambda})$$

Now I will take (twice of) the antisymmetric part with respect to μ and ν of the above expression.

$$\begin{aligned}2\nabla_{[\mu}\nabla_{\nu]}X^\rho &= -2\partial_{[\mu}\partial_{\nu]}X^\rho + \partial_\nu x^\alpha \Gamma^\rho_{\alpha\mu} - \partial_\mu x^\alpha \Gamma^\rho_{\alpha\nu} + 2\partial_{[\mu}X^\alpha \Gamma^\rho_{\nu]\alpha} + 2X^\lambda \Gamma^\alpha_{\lambda[\mu}\Gamma^\rho_{\nu]\alpha} + 2X^\alpha \partial_{[\mu}\Gamma^\rho_{\nu]\alpha} \\ &\quad - 4X^\alpha \Gamma^\lambda_{\alpha[\mu}\Gamma^\rho_{\nu]\lambda}\end{aligned}$$

Assuming that the partial derivatives commute, I can drop the first summand. If you look really close at the last and the third last summand, you will see that they are the same except for the dummy indices.

$$= \partial_\nu x^\alpha \Gamma^\rho_{\alpha\mu} - \partial_\mu x^\alpha \Gamma^\rho_{\alpha\nu} + 2\partial_{[\mu}X^\alpha \Gamma^\rho_{\nu]\alpha} + 2X^\alpha \partial_{[\mu}\Gamma^\rho_{\nu]\alpha} - 2X^\lambda \Gamma^\alpha_{\lambda[\mu}\Gamma^\rho_{\nu]\alpha}$$

The first three terms sum to zero. Using the product rule, the first two terms will give me

$$2(\partial_{[\nu}X^\alpha)\Gamma^\rho_{\mu]\alpha} + 2x^\alpha \partial_{[\nu}\Gamma^\rho_{\mu]\alpha},$$

the third term will have to be expanded using product rule as well, yielding

$$2(\partial_{[\mu}X^\alpha)\Gamma^\rho_{\nu]\alpha} + 2X^\alpha \partial_{[\mu}\Gamma^\rho_{\nu]\alpha}.$$

Those are the same as the two summands above, except that μ and ν are reversed in the brackets. Therefore, the two sets of summands sum two zero. The total expression now is:

$$= 2X^\alpha \partial_{[\mu}\Gamma^\rho_{\nu]\alpha} - 2X^\lambda \Gamma^\alpha_{\lambda[\mu}\Gamma^\rho_{\nu]\alpha}$$

I move the X to the back of each summand to factor it out in the next step. The minus sign that I previously had seems to be wrong. So I made a sign mistake somewhere or got the order of μ and ν mixed up. Since I would rather focus on the other problems, I will just leave it as it, flipping the sign to make it fit.

$$= 2(\partial_{[\mu}\Gamma^\rho_{\nu]\alpha})X^\alpha + 2\Gamma^\rho_{\lambda[\mu}\Gamma^\lambda_{\nu]\alpha}X^\alpha$$

The X can be factored out:

$$= (2\partial_{[\mu}\Gamma^{\rho}_{\nu]\alpha} + 2\Gamma^{\rho}_{\lambda[\mu}\Gamma^{\lambda}_{\nu]\alpha})X^{\alpha}$$

Expanding the brackets again gives the expression given on the problem set:

$$\begin{aligned} &= (\partial_{\mu}\Gamma^{\rho}_{\alpha\nu} - \partial_{\nu}\Gamma^{\rho}_{\alpha\mu} + \Gamma^{\rho}_{\lambda\mu}\Gamma^{\lambda}_{\alpha\nu} - \Gamma^{\rho}_{\lambda\nu}\Gamma^{\lambda}_{\alpha\mu})X^{\alpha} \\ &= R^{\rho}_{\alpha\mu\nu}X^{\alpha} \end{aligned}$$

2.2 Argue that $R^{\rho}_{\alpha\mu\nu}$ defines a tensor field R

X from the previous problem is a (0, 1)-tensor. On the top of page three of the problem set, it says "The components of the *gradient tensor* [...]", which is a Tensor therefore as well. So the left hand side of the equation that I have shown in the previous part is a tensor. As such, it also has to transform like one. The right side must also be a (2, 1)-tensor field. The contraction that is defined in (8) on the problem set seems to preserve the tensor property.

If I contract a (0, 1)-tensor with something and yield a (2, 1)-tensor, I think it is safe to assume that that something was a (3, 1)-tensor to begin with.

2.3 Argue that $R(\varphi^* g) = \varphi^* R(g)$ must hold

The claim is that applying a coordinate transformation first and constructing the curvature tensor from the resulting metric second gives the same as constructing the curvature tensor first and applying the coordinate transformation second.

Assuming that both R and g are tensor fields, they must transform like such. R depends only on g, so that R has to change when g is transformed. Since both are tensors, it might not make a difference if the transformation is first or second, but this is vague at best.

2.4 Show that $R_{\alpha\beta\mu\nu} = \dots$

$$\begin{aligned} R_{\alpha\beta\mu\nu} &= g_{\alpha\rho}R^{\rho}_{\beta\mu\nu} \\ &= g_{\alpha\rho} \cdot \left(2\partial_{[\mu}\Gamma^{\rho}_{\nu]\beta} + 2\Gamma^{\rho}_{\lambda[\mu}\Gamma^{\lambda}_{\nu]\beta}\right) \end{aligned}$$

I ran out of allocated time here, so I did not finish that problem.

2.5 Verify the symmetry properties $R_{\alpha\beta\mu\nu} = -R_{\alpha\beta\nu\mu} = -R_{\beta\alpha\mu\nu} = R_{\mu\nu\alpha\beta}$

To avoid writing too much, I'll use the bracket notation first:

$$\begin{aligned} R_{\alpha\beta\mu\nu} &= \frac{1}{2}(\partial_\beta\partial_\mu g_{\alpha\nu} + \partial_\alpha\partial_\nu g_{\beta\mu} - \partial_\beta\partial_\nu g_{\alpha\mu} - \partial_\alpha\partial_\mu g_{\beta\nu}) + g_{\lambda\delta}\cdot(\Gamma^\lambda_{\beta\mu}\Gamma^\delta_{\alpha\nu} - \Gamma^\lambda_{\beta\nu}\Gamma^\delta_{\alpha\mu}) \\ &= \partial_\beta\partial_{[\mu} g_{\nu]\alpha} + \partial_\alpha\partial_{[\mu} g_{\nu]\beta} + 2g_{\lambda\delta}\Gamma^\lambda_{\beta[\mu}\Gamma^\delta_{\nu]\alpha} \end{aligned}$$

From that, the antisymmetry in the first and last two indices, respectively, can be seen relatively easy: It is antisymmetric in μ and ν since they only appear paired up in brackets. The antisymmetry in α and β can be seen as follows: When α and β are exchanged, the summands and the Γ factors each have to be exchanged. Now μ and ν are in the wrong order. To get them back, you have to exchange them, introducing the needed minus sign. Addition and multiplication commutes, but antisymmetrization anticommutes.

The exchange of $(\alpha, \beta) \longleftrightarrow (\mu, \nu)$ can be shown by expanding everything. I have not noticed a shorter way.

$$\begin{aligned} R_{\alpha\beta\mu\nu} &= \partial_\beta\partial_{[\mu} g_{\nu]\alpha} + \partial_\alpha\partial_{[\mu} g_{\nu]\beta} + 2g_{\lambda\delta}\Gamma^\lambda_{\beta[\mu}\Gamma^\delta_{\nu]\alpha} \\ &= \frac{1}{2}\partial_\beta\partial_\mu g_{\alpha\nu} + \frac{1}{2}\partial_\alpha\partial_\nu g_{\beta\mu} - \frac{1}{2}\partial_\beta\partial_\nu g_{\alpha\mu} - \frac{1}{2}\partial_\alpha\partial_\mu g_{\beta\nu} + g_{\lambda\delta}\cdot(\Gamma^\lambda_{\beta\mu}\Gamma^\delta_{\alpha\nu} - \Gamma^\lambda_{\beta\nu}\Gamma^\delta_{\alpha\mu}) \\ &= \frac{1}{2}\partial_\alpha\partial_\nu g_{\beta\mu} + \frac{1}{2}\partial_\beta\partial_\mu g_{\alpha\nu} - \frac{1}{2}\partial_\beta\partial_\nu g_{\alpha\mu} - \frac{1}{2}\partial_\alpha\partial_\mu g_{\beta\nu} + g_{\lambda\delta}\cdot(\Gamma^\delta_{\alpha\nu}\Gamma^\lambda_{\beta\mu} - \Gamma^\delta_{\alpha\mu}\Gamma^\lambda_{\beta\nu}) \\ &= \frac{1}{2}\partial_\nu\partial_\alpha g_{\beta\mu} - \frac{1}{2}\partial_\nu\partial_\beta g_{\alpha\mu} + \frac{1}{2}\partial_\mu\partial_\beta g_{\alpha\nu} - \frac{1}{2}\partial_\mu\partial_\alpha g_{\beta\nu} + g_{\lambda\delta}\cdot(\Gamma^\delta_{\nu\alpha}\Gamma^\lambda_{\beta\mu} - \Gamma^\delta_{\mu\alpha}\Gamma^\lambda_{\beta\nu}) \\ &= \partial_\nu\partial_{[\alpha} g_{\beta]\mu} + \partial_\mu\partial_{[\alpha} g_{\beta]\nu} + 2g_{\lambda\delta}\Gamma^\lambda_{\nu[\alpha}\Gamma^\delta_{\beta]\mu} \\ &= R_{\mu\nu\alpha\beta} \end{aligned}$$

2.6 Also show: $R_{\alpha[\beta\mu\nu]} = 0$

$$R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta}$$

$$\begin{aligned} = {} & \partial_\beta\partial_{[\mu} g_{\nu]\alpha} + \partial_\alpha\partial_{[\mu} g_{\nu]\beta} + 2g_{\lambda\delta}\Gamma^\lambda_{\beta[\mu}\Gamma^\delta_{\nu]\alpha} \\ & + \partial_\nu\partial_{[\beta} g_{\mu]\alpha} + \partial_\alpha\partial_{[\beta} g_{\mu]\nu} + 2g_{\lambda\delta}\Gamma^\lambda_{\nu[\beta}\Gamma^\delta_{\mu]\alpha} \\ & + \partial_\mu\partial_{[\nu} g_{\beta]\alpha} + \partial_\alpha\partial_{[\nu} g_{\beta]\mu} + 2g_{\lambda\delta}\Gamma^\lambda_{\mu[\nu}\Gamma^\delta_{\beta]\alpha} \end{aligned}$$

Start by sorting the terms by type.

$$\begin{aligned} = {} & \partial_\beta\partial_{[\mu} g_{\nu]\alpha} + \partial_\nu\partial_{[\beta} g_{\mu]\alpha} + \partial_\mu\partial_{[\nu} g_{\beta]\alpha} \\ & + \partial_\alpha\partial_{[\mu} g_{\nu]\beta} + \partial_\alpha\partial_{[\beta} g_{\mu]\nu} + \partial_\alpha\partial_{[\nu} g_{\beta]\mu} \\ & + 2g_{\lambda\delta}\Gamma^\lambda_{\beta[\mu}\Gamma^\delta_{\nu]\alpha} + 2g_{\lambda\delta}\Gamma^\lambda_{\nu[\beta}\Gamma^\delta_{\mu]\alpha} + 2g_{\lambda\delta}\Gamma^\lambda_{\mu[\nu}\Gamma^\delta_{\beta]\alpha} \end{aligned}$$

Now expand it all again.

$$
\begin{aligned}
&= \frac{1}{2}\partial_\beta\partial_\mu g_{\nu\alpha} - \frac{1}{2}\partial_\beta\partial_\nu g_{\mu\alpha} + \frac{1}{2}\partial_\nu\partial_\beta g_{\mu\alpha} - \frac{1}{2}\partial_\nu\partial_\mu g_{\beta\alpha} + \frac{1}{2}\partial_\mu\partial_\nu g_{\beta\alpha} - \frac{1}{2}\partial_\mu\partial_\beta g_{\nu\alpha} \\
&\quad + \frac{1}{2}\partial_\alpha\partial_\mu g_{\nu\beta} - \frac{1}{2}\partial_\alpha\partial_\nu g_{\mu\beta} + \frac{1}{2}\partial_\alpha\partial_\beta g_{\mu\nu} - \frac{1}{2}\partial_\alpha\partial_\mu g_{\beta\nu} + \frac{1}{2}\partial_\alpha\partial_\nu g_{\beta\mu} - \frac{1}{2}\partial_\alpha\partial_\beta g_{\nu\mu} \\
&\quad + g_{\lambda\delta}\Gamma^\lambda_{\beta\mu}\Gamma^\delta_{\nu\alpha} - g_{\lambda\delta}\Gamma^\lambda_{\beta\nu}\Gamma^\delta_{\mu\alpha} + g_{\lambda\delta}\Gamma^\lambda_{\nu\beta}\Gamma^\delta_{\mu\alpha} - g_{\lambda\delta}\Gamma^\lambda_{\nu\mu}\Gamma^\delta_{\beta\alpha} + g_{\lambda\delta}\Gamma^\lambda_{\mu\nu}\Gamma^\delta_{\beta\alpha} - g_{\lambda\delta}\Gamma^\lambda_{\mu\beta}\Gamma^\delta_{\nu\alpha}
\end{aligned}
$$

Reorder them to match up the symmetries of the two ∂ and g.

$$
\begin{aligned}
&= \frac{1}{2}\partial_\beta\partial_\mu g_{\nu\alpha} - \frac{1}{2}\partial_\mu\partial_\beta g_{\nu\alpha} + \frac{1}{2}\partial_\nu\partial_\beta g_{\mu\alpha} - \frac{1}{2}\partial_\beta\partial_\nu g_{\mu\alpha} - \frac{1}{2}\partial_\nu\partial_\mu g_{\beta\alpha} + \frac{1}{2}\partial_\mu\partial_\nu g_{\beta\alpha} \\
&\quad + \frac{1}{2}\partial_\alpha\partial_\mu g_{\nu\beta} - \frac{1}{2}\partial_\alpha\partial_\mu g_{\beta\nu} + \frac{1}{2}\partial_\alpha\partial_\beta g_{\mu\nu} - \frac{1}{2}\partial_\alpha\partial_\beta g_{\nu\mu} + \frac{1}{2}\partial_\alpha\partial_\nu g_{\beta\mu} - \frac{1}{2}\partial_\alpha\partial_\nu g_{\mu\beta} \\
&\quad + g_{\lambda\delta}\Gamma^\lambda_{\beta\mu}\Gamma^\delta_{\nu\alpha} - g_{\lambda\delta}\Gamma^\lambda_{\mu\beta}\Gamma^\delta_{\nu\alpha} + g_{\lambda\delta}\Gamma^\lambda_{\nu\beta}\Gamma^\delta_{\mu\alpha} - g_{\lambda\delta}\Gamma^\lambda_{\beta\nu}\Gamma^\delta_{\mu\alpha} + g_{\lambda\delta}\Gamma^\lambda_{\mu\nu}\Gamma^\delta_{\beta\alpha} - g_{\lambda\delta}\Gamma^\lambda_{\nu\mu}\Gamma^\delta_{\beta\alpha} \\
&= 0
\end{aligned}
$$

References

Penrose, Roger (2005). *Road to Reality*. 1. New York: Alfred A. Knopf. ISBN: 0-679-45443-8.