Disclaimer

This is a reviewed problem set for the module physics751.

This problem set was reviewed by a tutor. *This does not mean that it is a perfect solution. Neither I or the tutor imply that there are no further mistakes in this document.*

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[disclaimer]

physics751 – Group Theory

Problem Set 12

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1 Momentum operator

I copy the gradient from (Wikipedia 2014):

$$\nabla = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{e}_\phi \frac{1}{r \sin(\theta)} \frac{\partial}{\partial \phi}.$$

Then with $\boldsymbol{r} = r\hat{e}_r$ I can form the cross product, which is linear in its two arguments:

$$\begin{aligned} \boldsymbol{r} \times \nabla &= r\hat{e}_r \times \left[\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{e}_\phi \frac{1}{r \sin(\theta)} \frac{\partial}{\partial \phi} \right]. \\ &= r\hat{e}_r \times \hat{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + r\hat{e}_r \times \hat{e}_\phi \frac{1}{r \sin(\theta)} \frac{\partial}{\partial \phi} \end{aligned}$$

Cancel r.

$$= \hat{e}_r \times \hat{e}_\theta \frac{\partial}{\partial \theta} + \hat{e}_r \times \hat{e}_\phi \frac{1}{\sin(\theta)} \frac{\partial}{\partial \phi}$$

The vectors $\{\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi\}$ form a right-handed orthonormal basis. That means that those cross products are simply cyclic.

$$= \hat{e}_\phi \frac{\partial}{\partial \theta} - \hat{e}_\theta \frac{1}{\sin(\theta)} \frac{\partial}{\partial \phi}$$

✓

2 Laplacian in three dimensions

$$
\begin{aligned}
L^2 &= -\left[\hat{\mathbf{e}}_\phi \frac{\partial}{\partial\theta} - \hat{\mathbf{e}}_\theta \frac{1}{\sin(\theta)}\frac{\partial}{\partial\phi}\right]^2 \\
&= -\left[\hat{\mathbf{e}}_\phi \frac{\partial}{\partial\theta} - \hat{\mathbf{e}}_\theta \frac{1}{\sin(\theta)}\frac{\partial}{\partial\phi}\right]\left[\hat{\mathbf{e}}_\phi \frac{\partial}{\partial\theta} - \hat{\mathbf{e}}_\theta \frac{1}{\sin(\theta)}\frac{\partial}{\partial\phi}\right] \\
&= -\left[\hat{\mathbf{e}}_\phi \frac{\partial}{\partial\theta}\hat{\mathbf{e}}_\phi \frac{\partial}{\partial\theta} - \hat{\mathbf{e}}_\phi \frac{\partial}{\partial\theta}\hat{\mathbf{e}}_\theta \frac{1}{\sin(\theta)}\frac{\partial}{\partial\phi} - \hat{\mathbf{e}}_\theta \frac{1}{\sin(\theta)}\frac{\partial}{\partial\phi}\hat{\mathbf{e}}_\phi \frac{\partial}{\partial\theta} + \hat{\mathbf{e}}_\theta \frac{1}{\sin(\theta)}\frac{\partial}{\partial\phi}\hat{\mathbf{e}}_\theta \frac{1}{\sin(\theta)}\frac{\partial}{\partial\phi}\right]
\end{aligned}
$$

Now I have apply the product rule to get even more terms. I will not do them all at once since the page is not wide enough, even in landscape. The first two look like this when expanded:

$$
= -\left[\hat{\mathbf{e}}_\phi \left[\frac{\partial}{\partial\theta}\hat{\mathbf{e}}_\phi\right]\frac{\partial}{\partial\theta} + \hat{\mathbf{e}}_\phi\hat{\mathbf{e}}_\phi \frac{\partial^2}{\partial\theta^2} - \hat{\mathbf{e}}_\phi\hat{\mathbf{e}}_\theta \frac{\partial}{\partial\theta}\frac{1}{\sin(\theta)}\frac{\partial}{\partial\phi} - \hat{\mathbf{e}}_\phi\left[\frac{\partial}{\partial\theta}\hat{\mathbf{e}}_\theta\right]\frac{1}{\sin(\theta)}\frac{\partial}{\partial\phi} - \hat{\mathbf{e}}_\theta \frac{1}{\sin(\theta)}\frac{\partial}{\partial\phi}\hat{\mathbf{e}}_\phi \frac{\partial}{\partial\theta} + \hat{\mathbf{e}}_\theta \frac{1}{\sin(\theta)}\frac{\partial}{\partial\phi}\hat{\mathbf{e}}_\theta \frac{1}{\sin(\theta)}\frac{\partial}{\partial\phi}\right]
$$

The first term is zero since $\hat{\mathbf{e}}_\phi$ has no dependence on θ. The second one can be simplified since the vectors are unit vectors. The third term is zero because of the orthogonality of the basis vectors.

$$
= -\left[\frac{\partial^2}{\partial\theta^2} - \hat{\mathbf{e}}_\phi\left[\frac{\partial}{\partial\theta}\hat{\mathbf{e}}_\theta\right]\frac{1}{\sin(\theta)}\frac{\partial}{\partial\phi} - \hat{\mathbf{e}}_\theta \frac{1}{\sin(\theta)}\frac{\partial}{\partial\phi}\hat{\mathbf{e}}_\phi \frac{\partial}{\partial\theta} + \hat{\mathbf{e}}_\theta \frac{1}{\sin(\theta)}\frac{\partial}{\partial\phi}\hat{\mathbf{e}}_\theta \frac{1}{\sin(\theta)}\frac{\partial}{\partial\phi}\right]
$$

Now there is a bit space left and I will expand the next term.

$$
= -\left[\frac{\partial^2}{\partial\theta^2} - \hat{\mathbf{e}}_\phi\left[\frac{\partial}{\partial\theta}\hat{\mathbf{e}}_\theta\right]\frac{1}{\sin(\theta)}\frac{\partial}{\partial\phi} - \hat{\mathbf{e}}_\theta \frac{1}{\sin(\theta)}\left[\frac{\partial}{\partial\phi}\hat{\mathbf{e}}_\phi\right]\frac{\partial}{\partial\theta} - \hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_\phi \frac{1}{\sin(\theta)}\frac{\partial}{\partial\phi}\frac{\partial}{\partial\theta} + \hat{\mathbf{e}}_\theta \frac{1}{\sin(\theta)}\frac{\partial}{\partial\phi}\hat{\mathbf{e}}_\theta \frac{1}{\sin(\theta)}\frac{\partial}{\partial\phi}\right]
$$

Here, the first new term has to be evaluated a bit more closely, but the second new one is zero because of the orthogonality as well. Now I can expand the last term.

$$
= -\left[\frac{\partial^2}{\partial\theta^2} - \hat{\mathbf{e}}_\phi\left[\frac{\partial}{\partial\theta}\hat{\mathbf{e}}_\theta\right]\frac{1}{\sin(\theta)}\frac{\partial}{\partial\phi} - \hat{\mathbf{e}}_\theta \frac{1}{\sin(\theta)}\left[\frac{\partial}{\partial\phi}\hat{\mathbf{e}}_\phi\right]\frac{\partial}{\partial\theta} + \hat{\mathbf{e}}_\theta \frac{1}{\sin(\theta)^2}\left[\frac{\partial}{\partial\phi}\hat{\mathbf{e}}_\theta\right]\frac{\partial}{\partial\phi} + \hat{\mathbf{e}}_\theta \frac{1}{\sin(\theta)^2}\left[\frac{\partial}{\partial\phi}\hat{\mathbf{e}}_\theta\right]\frac{\partial}{\partial\phi} + \frac{1}{\sin(\theta)^2}\frac{\partial^2}{\partial\phi^2}\right]
$$

The last term can be simplified as well because of the normality. I expand the leading minus to get rid of that square bracket.

$$= -\frac{\partial^2}{\partial \theta^2} + \hat{\mathbf{e}}_\phi \left[\frac{\partial}{\partial \theta} \hat{\mathbf{e}}_\theta \right] \frac{1}{\sin(\theta)} \frac{\partial}{\partial \phi} + \hat{\mathbf{e}}_\theta \frac{1}{\sin(\theta)} \left[\frac{\partial}{\partial \phi} \hat{\mathbf{e}}_\phi \right] \frac{\partial}{\partial \theta} - \hat{\mathbf{e}}_\theta \frac{1}{\sin(\theta)} \left[\frac{\partial}{\partial \phi} \hat{\mathbf{e}}_\theta \right] \frac{1}{\sin(\theta)} \frac{\partial}{\partial \phi} - \frac{1}{\sin(\theta)^2} \frac{\partial^2}{\partial \phi^2}$$

I computed $\partial \hat{\mathbf{e}}_\theta / \partial \theta$ and it turns out to be $-\hat{\mathbf{e}}_r$. This is orthogonal to $\hat{\mathbf{e}}_\phi$. The second term is zero, therefore. The derivative $\partial \hat{\mathbf{e}}_\theta / \partial \phi$ gives $-\cos(\theta)\hat{\mathbf{e}}_\phi$ which is orthogonal to $\hat{\mathbf{e}}_\theta$. So the fourth term drops as well.

$$= -\frac{\partial^2}{\partial \theta^2} + \hat{\mathbf{e}}_\theta \frac{1}{\sin(\theta)} \left[\frac{\partial}{\partial \phi} \hat{\mathbf{e}}_\phi \right] \frac{\partial}{\partial \theta} - \frac{1}{\sin(\theta)^2} \frac{\partial^2}{\partial \phi^2}$$

The derivative $\partial \hat{\mathbf{e}}_\phi / \partial \phi$ gives something that is in the z-plane and not orthogonal to $\hat{\mathbf{e}}_\theta$. The scalar product will give a contribution of $-\cos(\theta)$ here.

$$= -\frac{\partial^2}{\partial \theta^2} - \cot(\theta) \frac{\partial}{\partial \theta} - \frac{1}{\sin(\theta)^2} \frac{\partial^2}{\partial \phi^2}$$

Using the product rule in reverse, this can be written more compactly.

$$= -\left[\frac{\partial^2}{\partial \theta^2} + \cot(\theta) \frac{\partial}{\partial \theta} \right] - \frac{1}{\sin(\theta)^2} \frac{\partial^2}{\partial \phi^2}$$

I insert a one into the first summand and expand the cotangent.

$$= -\left[\frac{1}{\sin(\theta)} \sin(\theta) \frac{\partial^2}{\partial \theta^2} + \frac{\cos(\theta)}{\sin(\theta)} \frac{\partial}{\partial \theta} \right] - \frac{1}{\sin(\theta)^2} \frac{\partial^2}{\partial \phi^2}$$

Then I factor out the cosecant.

$$= -\frac{1}{\sin(\theta)} \left[\sin(\theta) \frac{\partial^2}{\partial \theta^2} + \cos(\theta) \frac{\partial}{\partial \theta} \right] - \frac{1}{\sin(\theta)^2} \frac{\partial^2}{\partial \phi^2}$$

Write the cosine as the derivative of the sine.

$$= -\frac{1}{\sin(\theta)}\left[\sin(\theta)\frac{\partial^2}{\partial\theta^2} + \frac{\partial \sin(\theta)}{\partial\theta}\frac{\partial}{\partial\theta}\right] - \frac{1}{\sin(\theta)^2}\frac{\partial^2}{\partial\phi^2}$$

Then I factor out the last derivative.

$$= -\frac{1}{\sin(\theta)}\left[\sin(\theta)\frac{\partial}{\partial\theta} + \frac{\partial}{\partial\theta}\sin(\theta)\right]\frac{\partial}{\partial\theta} - \frac{1}{\sin(\theta)^2}\frac{\partial^2}{\partial\phi^2}$$

Now reverse the product rule.

$$= -\frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\sin(\theta)\frac{\partial}{\partial\theta} - \frac{1}{\sin(\theta)^2}\frac{\partial^2}{\partial\phi^2}$$

very nice, if this would have been an exam task you would have started here

That is the expression given on the problem set.

Therefore the Laplacian can be written as

$$\Delta = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} - \frac{L^2}{r^2}.$$

✓

3 Constrain to unit sphere

The constraining to the sphere means the constraint $r - 1 = 0$ is added. Therefore, r is not a free coordinate of the system any more, only θ and ϕ are. The Hamiltonian then becomes:

$$H = \frac{L^2}{2m}.$$

The eigenfunctions are the spherical harmonics Y_{lm} with L^2 eigenvalue $l[l+1]$. The eigenenergies therefore are:

$$E_l = \frac{l[l+1]}{2m}.$$

In the regular hydrogen atom, there arises the need for a quantum number n in order for the radial part to be normalizable. This is not the case here, so there is no need for n and l and m are the only quantum numbers needed. The degeneracy therefore is just $2l + 1$.

✓

4 Laplacian in two dimensions

This is more of less the same thing as in the first homework problem. There are just less components and more vanishing derivatives of the basis vectors. The gradient is

$$\nabla = \hat{e}_\rho \frac{\partial}{\partial \rho} + \hat{e}_\phi \frac{1}{\rho} \frac{\partial}{\partial \phi}.$$

Then the cross product in the angular momentum operator is

$$\rho \times \nabla = \rho \hat{e}_\rho \times \hat{e}_\phi \frac{1}{\rho} \frac{\partial}{\partial \phi} = \hat{e}_z \frac{\partial}{\partial \phi}.$$

The square is simple because $\hat{e}_z$ does not depend on ϕ at all. Therefore

$$L^2 = -\frac{\partial^2}{\partial \phi^2}.$$

The Hamiltonian with the interaction is then given by

$$H = \frac{L^2}{2m} - \frac{e^2}{\rho}.$$

✓

5 Degeneracy

The solutions seem to be $\psi(\phi)$ only. A product ansatz like $R(\rho)\Phi(\phi)$ will probably work. The angular part is most likely given by $\Phi(\phi) = \exp(il\phi)$. The $R(\rho)$ will probably have to have an n such that the polynomial ends somewhere and $\lim_{\rho \to \infty} R(\rho) = 0$ is given.

Since there is just one angle, I am not sure whether there is a degeneracy after all.

Since $-l$ and $+l$ are both eigenvalues to the eigenfunction $\exp(il\phi)$, the degeneracy is simply 2!!

References

Wikipedia (2014). *Del in cylindrical and spherical coordinates*. URL: http://en.wikipedia.org/w/index.php?title=Del_in_cylindrical_and_spherical_coordinates&oldid=625329099.