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This is a reviewed problem set for the module physics751.

This problem set was reviewed by a tutor. *This does not mean that it is a perfect solution. Neither I or the tutor imply that there are no further mistakes in this document.*

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[disclaimer]

physics751 – Group Theory

Problem Set 7

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Group 1 – Patrick Matuschek

1 The character table of D_4

1.1 Character table

Problem statement

Compute the character table for D_4.

Hint: Make use of the first orthogonality theorem for characters.

The group D_4 is given by

$$D_4 = \{e, c, c^2, c^3, b, bc, bc^2, bc^3\}.$$

The conjugacy classes are

$$c_1 = \{e\}, \quad c_2 = \{c, c^3\}, \quad c_3 = \{c^2\}, \quad c_4 = \{b, bc^2\}, \quad c_5 = \{bc, bc^3\}.$$

There are five classes, therefore the are five irreducible representations of this group. Their dimensions must be

$$8 = 1 + 1 + 1 + 1 + 2^2.$$

The dimensions alone give the χ_1, since the class c_1 is the one that contains the identity. The first represenation, Γ_1 is always the trivial one, where everything is mapped to 1. Therefore, the characters for all classes are 1. I summarize this information in the following character table:

	χ_1	χ_2	χ_3	χ_4	χ_5
Γ_1	1	1	1	1	1
Γ_2	1				
Γ_3	1				
Γ_4	1				
Γ_5	2				

In the one dimensional representations, where the group elements get mapped to a scalar, the character is the representation element itself. Together with the way the group product is mapped to the matrix multiplication, I get

$$\chi_2^{\Gamma_2} = \chi^{\Gamma_2}(c) = \chi^{\Gamma_2}(c^3) = \chi^{\Gamma_2}(c)\chi^{\Gamma_2}(c^2).$$

This implies

$$\chi^{\Gamma_2}(c^2) = 1.$$

That works in every one dimensional representation, so Γ_1 to Γ_4. This gives more entries in the table. The orthonormality relation for the columns gives me the $\chi_3^{\Gamma_5}$. This has to be ± 2. Since the columns have to be orthogonal, only -2 is a viable option.

	χ_1	χ_2	χ_3	χ_4	χ_5
Γ_1	1	1	1	1	1
Γ_2	1		1		
Γ_3	1		1		
Γ_4	1		1		
Γ_5	2		-2		

The row sum of squared characters (times number of elements on the class) in the last row is already 8. That means that all other characters have to be zero.

b is its own inverse. Its character in the one dimensional representations can only be ± 1 then. For the classes c_2 and c_5, the order of the elements is 4. Since I assume a real carrier space (can I always do that?), the only options are ± 1 as well.

The table looks like this now:

	χ_1	χ_2	χ_3	χ_4	χ_5
Γ_1	1	1	1	1	1
Γ_2	1	± 1	1	± 1	± 1
Γ_3	1	± 1	1	± 1	± 1
Γ_4	1	± 1	1	± 1	± 1
Γ_5	2	0	-2	0	0

The squared row and column sums all check out to be 8, so the magnitudes of all characters seems to be fine. Now the signs have to be chosen in a way that all the rows and columns are orthonormal. I chose the signs stepwise such that the new row is orthogonal to the ones that are fixed until then. This is the table that I got:

	χ_1	χ_2	χ_3	χ_4	χ_5
Γ_1	1	1	1	1	1
Γ_2	1	1	1	−1	−1
Γ_3	1	−1	1	1	−1
Γ_4	1	−1	1	−1	1
Γ_5	2	0	−2	0	0

As far as I can tell, all the rows and columns are orthogonal and sum to 8 in the appropriate manner.

Side question

Maybe there is a more systematic way to chose the signs. I set up a system of equations for all the scalar products that have to vanish. Since this was a non linear system of equations I did not know of any systematic approach. Is there one?

No, I don't think there is. Of course you always need to make sure you really get a representation.

1.2 Representation product

Problem statement

Let $D^{(5)}$ denote the two-dimensional irreducible representation of D_4. Check whether the product $D^{(5)} \otimes D^{(5)}$ is reducible and calculate the Clebsch-Gordan decomposition if necessary.

The expansion works like this:

$$D^{\Gamma_5 \times \Gamma_5} = D^{\Gamma_5} \otimes D^{\Gamma_5} = \sum_i a_i D^{\Gamma_i}$$

The expansion coefficients can be computed with

$$a_i = \langle \chi^{\Gamma_i}, \chi^{\Gamma_5} \chi^{\Gamma_5} \rangle.$$

The character vectors are given by:

$$\begin{aligned}
\chi^{\Gamma_1} &= \begin{pmatrix}1 & 1 & 1 & 1 & 1\end{pmatrix} \\
\chi^{\Gamma_2} &= \begin{pmatrix}1 & 1 & 1 & -1 & -1\end{pmatrix} \\
\chi^{\Gamma_3} &= \begin{pmatrix}1 & -1 & 1 & 1 & -1\end{pmatrix} \\
\chi^{\Gamma_4} &= \begin{pmatrix}1 & -1 & 1 & -1 & 1\end{pmatrix} \\
\chi^{\Gamma_5} &= \begin{pmatrix}2 & 0 & -2 & 0 & 0\end{pmatrix}
\end{aligned}$$

The order of the group has to be divided out, the number of elements on each class has to be taken into account.

All that is left is to compute the scalar products. The second term is

$$\chi^{\Gamma_5}\chi^{\Gamma_5} = \begin{pmatrix}4 & 0 & 4 & 0 & 0\end{pmatrix}.$$

Now I can compute the coefficients.

$$\begin{aligned}
a_1 &= \langle \chi^{\Gamma_1}, \chi^{\Gamma_5}\chi^{\Gamma_5} \rangle \\
&= \frac{1}{8}[1 \cdot 1 \cdot 4 + 1 \cdot 1 \cdot 4] \\
&= 1
\end{aligned}$$

Since the character vectors do not differ from the first one, the scalar products are all the same.

$$\begin{aligned}
a_2 &= 1 \\
a_3 &= 1 \\
a_4 &= 1 \\
a_5 &= \langle \chi^{\Gamma_1}, \chi^{\Gamma_5}\chi^{\Gamma_5} \rangle \\
&= \frac{1}{8}[1 \cdot 2 \cdot 4 + 1 \cdot [-2] \cdot 4] \\
&= 0
\end{aligned}$$

So I end up with

$$D^{\Gamma_5 \times \Gamma_5} = D^{\Gamma_5} \otimes D^{\Gamma_5} = D^{\Gamma_1} + D^{\Gamma_2} + D^{\Gamma_3} + D^{\Gamma_4}.$$

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very good