Disclaimer

This is a reviewed problem set for the module physics751.

This problem set was reviewed by a tutor. *This does not mean that it is a perfect solution. Neither I or the tutor imply that there are no further mistakes in this document.*

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physics751 – Group Theory

Problem Set 6

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1 A representation of D_3 on polynomials

Problem statement

Consider the 6-dimensional space of polynomials of degree 2 in two variables (x, y). Assuming that x and y transform under the dihedral group D_3 as the coordinates of a 2-vector with $\boldsymbol{D}^2(b)$, $\boldsymbol{D}^2(c)$ and $\boldsymbol{D}^2(c^2)$, we obtain a 6-dimensional representation of D_3 on the above function space. Write down the matrices of this representation …

The vectors of the six dimensional vector space are the various $f(x, y)$. The components of the vectors are the a_i. Addition of multiple vectors f is simply addition of the vectors $\boldsymbol{a}$. Exactly as I did on the last problem set, I write everything in terms of vectors and matrices. The polynomial can be written in components with the basis vectors $\hat{\boldsymbol{e}}$ out of $\{1, x, y, x^2, xy, y^2\}$ like so:

$$f(x, y) = a^i \hat{\boldsymbol{e}}_i.$$

Now I can find the six dimensional representation such that it fulfills this:

$$f\left(\boldsymbol{D}^2(g)\begin{pmatrix} x \\ y \end{pmatrix}\right) \stackrel{!}{=} \left[\boldsymbol{D}^6(g)\boldsymbol{a}\right]^i \hat{\boldsymbol{e}}_i.$$

Well, this is the same wrong transformation as I did on the last problem set. Those should be D^{-1} there. ✓

I will start with the first one:

$$\begin{aligned} f\left(\boldsymbol{D}^2(c)\begin{pmatrix} x \\ y \end{pmatrix}\right) &= f(x, -y) \\ &= a_1 + a_2 x - a_3 y + a_4 x^2 - a_5 xy + a_6 y^2 \end{aligned}$$

I really should know that from GR :-/

Now I can write this as a scalar product of a transformed vector:

$$= \begin{pmatrix} a_1 \\ a_2 \\ -a_3 \\ a_4 \\ -a_5 \\ a_6 \end{pmatrix} \begin{pmatrix} 1 \\ x \\ y \\ x^2 \\ xy \\ y^2 \end{pmatrix}.$$

← this is the wrong way to view this
e.g. x is allready a basis vector, so
$a_1 + a_2 x + a_3 y + \ldots$ and $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \end{pmatrix}$ are just
two different ways of writing f

The matrix that has transformed $\boldsymbol{a}$ can be read off. To simplify, I omit all the zeros in the matrices.

$$= \left[\begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & -1 & & & \\ & & & 1 & & \\ & & & & -1 & \\ & & & & & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix} \right] \begin{pmatrix} 1 \\ x \\ y \\ x^2 \\ xy \\ y^2 \end{pmatrix}$$

From this, I can recover:

$$D^6(b) = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & -1 & & & \\ & & & 1 & & \\ & & & & -1 & \\ & & & & & 1 \end{pmatrix}.$$ ✓

The next one is very similar, except that there are more terms.

$$f\left(\boldsymbol{D}^2(c)\begin{pmatrix} x \\ y \end{pmatrix}\right) = f\left(-\frac{1}{2}x - \frac{\sqrt{3}}{2}y, \frac{\sqrt{3}}{2}x - \frac{1}{2}d\right)$$

Expanding this gives a lot of terms:

$$\begin{aligned} &= a_1 + a_2\left[-\frac{1}{2}x - \frac{\sqrt{3}}{2}y\right] + a_3\left[\frac{\sqrt{3}}{2}x - \frac{1}{2}y\right] + a_4\left[-\frac{1}{2}x - \frac{\sqrt{3}}{2}y\right]^2 \\ &\quad + a_5\left[-\frac{1}{2}x - \frac{\sqrt{3}}{2}y\right]\left[\frac{\sqrt{3}}{2}x - \frac{1}{2}y\right] + a_6\left[\frac{\sqrt{3}}{2}x - \frac{1}{2}y\right]^2. \end{aligned}$$

The next step is to expand all the terms and regroup them with respect to the basis vectors. I omit typesetting the expansion here and skip right to the regrouping.

$$\begin{aligned} &= a_1 + x\left[-\frac{1}{2}a_2 + \frac{\sqrt{3}}{2}a_3\right] + y\left[-\frac{\sqrt{3}}{2}a_2 - \frac{1}{2}a_3\right] + x^2\left[\frac{1}{4}a_4 - \frac{\sqrt{3}}{4}a_5 + \frac{3}{4}a_6\right] \\ &\quad + xy\left[\frac{3}{2}a_4 - \frac{1}{2}a_5 - \frac{\sqrt{3}}{2}a_6\right] + y^2\left[\frac{3}{4}a_4 + \frac{\sqrt{3}}{4}a_5 + \frac{1}{4}a_6\right] \end{aligned}$$

The matrix can be read off here rather directly. I obtain

$$\boldsymbol{D}^6(c) = \begin{pmatrix} 1 & & & & & \\ & -\frac{1}{2} & \frac{\sqrt{3}}{2} & & & \\ & -\frac{\sqrt{3}}{2} & -\frac{1}{2} & & & \\ & & & \frac{1}{4} & -\frac{\sqrt{3}}{4} & \frac{3}{4} \\ & & & \frac{3}{4} & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ & & & \frac{3}{4} & \frac{\sqrt{3}}{4} & \frac{1}{4} \end{pmatrix}. \quad (\checkmark)$$

The same can be done for the group element c^2, were I yield

$$\boldsymbol{D}^6(c^2) = \begin{pmatrix} 1 & & & & & \\ & -\frac{1}{2} & -\frac{\sqrt{3}}{2} & & & \\ & \frac{\sqrt{3}}{2} & \frac{1}{2} & & & \\ & & & \frac{1}{4} & \frac{\sqrt{3}}{4} & \frac{3}{4} \\ & & & -\frac{3}{4} & -1 & -\frac{\sqrt{3}}{2} \\ & & & \frac{3}{4} & \frac{\sqrt{3}}{4} & \frac{1}{4} \end{pmatrix}.$$

To be thorough, one should now test $[\boldsymbol{D}^6(b)]^2 = \mathbb{1}_6$ and $[\boldsymbol{D}^6(c)]^2 = \boldsymbol{D}^6(c^2)$ since a representation must behave like the group elements. and you should calculate $D^6(bc), D^6(bc^2)$

Problem statement

… and identify the invariant subspaces under D_3 and the corresponding irreducible representations.

There is a $\mathbb{R}^1$ subspace which does not do anything. In this concrete case it is just a constant summand on the functions, which is invariant under any rotation or reflection.

Then there is the two dimensional representation which seems to be the transpose of the $\boldsymbol{D}^6$. The fact that it is transposed seems a little strange, maybe I did something wrong in my derivation? The fact that they are almost equal is probably not too surprising, since there is only one irreducible representation with dimension two in this representation.

The last one is a three dimensional representation and it seems rather irreducible as well since the submatrix is dense.*

The three irreducible representation are the trivial one, the $\boldsymbol{D}^2$ that was given on the problem set transposed and the new three dimensional one that is in the bottom right corner of all the three matrices that I have derived.

* actually this one is reducible, since

$$a_4x^2 + a_5xy + a_6y^2 = \underbrace{(a_4+a_6)\frac{x^2+y^2}{2}}_{\text{this is a one-dim. invariant subspace}} + a_5xy + (a_4-a_6)\frac{x^2-y^2}{2}$$

$$\Rightarrow D(D_3) = D^{(1)}(D_3) \oplus D^{(1)}(D_3) \oplus D^{(2)}(D_3) \oplus D^{(2)}(D_3)$$