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This problem set is not reviewed by a tutor. This is just what I have turned in.

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[disclaimer]

physics751 – Group Theory

Problem Set 4

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Side question

During the lecture the regular representation of the symmetric group $\mathscr{S}_2$ was given. The multiplication table was written down:

$$T_{\mathscr{S}_2} = \begin{pmatrix} e & (12) \\ (12) & e \end{pmatrix}.$$

From there, the representation was given as

$$\boldsymbol{D}^{[\Gamma_\text{reg}]}(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{D}^{[\Gamma_\text{reg}]}((12)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The definition given in the lecture was, as far as I recall,

$$D^{[\Gamma_\text{reg}]}_{jk}(g_i) = \begin{cases} 1 & g_j g_k = g_i \\ 0 & \text{else} \end{cases}.$$

In words, this would mean that the representation matrices for a group element is a matrix that just had ones everywhere where that group element appears in the multiplication table, zeros everywhere else. This is fine with me.

Then the lecturer said that on some occasions the multiplication table has to be rearranged in a way such that the identity is on the diagonal such that $D(e) = \mathbb{1}$. The rearrangement theorem asserts that this rearrangement of rows and columns is possible. What concerns me is the fact that this rearranging might make the indexing of group elements ill-defined.

The definition of multiplication table of a group G with elements g_i that I remember is $T_{ij} := g_i g_j$. For this, there has so be something that maps the *set* of group elements into an *ordered tuple* such that g_i is always the same element. Since the order itself is arbitrary, I do not mind when a permutation $\pi \in \mathcal{S}_{|G|}$ is applied such that

$$G = \{g_i : i \leq |G|\} = \{g_{\pi(i)} : i \leq |G|\}$$

still holds.

We were all asked to derive the representation of $\mathcal{S}_3$ at home, so that is what I attempted.

This is my $\mathcal{S}_3$ and the order in the set is the order I use for my indexing with i:

$$\{(1), (12), (13), (23), (123), (321)\}.$$

So those are g_1 to g_6, in that order. Now I computed the multiplication table using the fast multiplication and the rearrangement theorem. This is the table that I got:

	(1)	(12)	(13)	(23)	(123)	(321)
(1)	(1)	(12)	(13)	(23)	(123)	(321)
(12)	(12)	(1)	(321)	(123)	(32)	(13)
(13)	(13)	(123)	(1)	(321)	(12)	(23)
(23)	(23)	(321)	(123)	(1)	(13)	(12)
(123)	(123)	(13)	(23)	(12)	(321)	(1)
(321)	(321)	(23)	(12)	(13)	(1)	(123)

To get the unity elements on the diagonal only, I had to exchange the last two rows. One could also exchange the last two columns, but not both. This is the changed table.

	(1)	(12)	(13)	(23)	**(123)**	**(321)**
(1)	(1)	(12)	(13)	(23)	(123)	(321)
(12)	(12)	(1)	(321)	(123)	(32)	(13)
(13)	(13)	(123)	(1)	(321)	(12)	(23)
(23)	(23)	(321)	(123)	(1)	(13)	(12)
(321)	(321)	(23)	(12)	(13)	(1)	(123)
(123)	(123)	(13)	(23)	(12)	(321)	(1)

At this point, I do not know that the meaning of g_5 and g_6 are supposed to be now.

Then I tried the more formal way and took the definition. Using the ordering that I defined above, $D(e)$ will not be $\mathbb{1}$. The only way I see to get it to work with the definition is to apply a permutation such that the definition is not

$$D_{jk}^{[\Gamma_{\text{reg}}]}(g_i) = \begin{cases} 1 & g_{\pi(j)} g_{\pi(k)} = g_{\pi(i)} \\ 0 & \text{else} \end{cases}$$

but something like

$$D_{jk}^{[\Gamma_{\text{reg}}]}(g_i) = \begin{cases} 1 & g_{\pi(j)} g_k = g_i \\ 0 & \text{else} \end{cases}$$

where only one element is permuted. To me, this looks like s_j does not need to be s_k even if $j = k$.

What is the correct way of doing this?

1 Pauli matrices and the quaternion group Q_4

Problem statement

Consider the group generated by $\mathrm{i}\sigma_1$ and $\mathrm{i}\sigma_2$, the so-called quaternion group Q_4.

1.1 Multiplication table

Problem statement

Write down the multiplication table of this group.

The generators of this group to derive are $\mathrm{i}\sigma_1$ and $\mathrm{i}\sigma_2$. The previously given multiplication law is

$$\sigma_i\sigma_j = \delta_{ij} + \mathrm{i}\epsilon_{ijk}\sigma_k.$$

Now with $g_1 := \mathrm{i}\sigma_1$ and $g_2 := \mathrm{i}\sigma_2$ we can write

$$g_1 g_2 = \mathrm{i}^2\sigma_1\sigma_2 = -\mathrm{i}\sigma_3.$$

I will call this element $g_3 := -\mathrm{i}\sigma_3$.

Since I know quaterions from (Penrose 2005, §11.1), I will now use his notation:

$$\mathbf{i} := g_1 = \mathrm{i}\sigma_1, \quad \mathbf{j} := g_2 = \mathrm{i}\sigma_2, \quad \mathbf{k} := g_3 = -\mathrm{i}\sigma_3.$$

The identity element e can be made up from $\mathbf{i}^4$ or $\mathbf{j}^4$ and $\mathbf{k}^4$. Therefore, the order of those elements is 4, by the way. These fulfill the equation that Hamilton wrote onto a stone of Dublin's Brougham

Bridge on 1843-10-16 (Penrose 2005, p. 198):

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1.$$

The last one is not trivial to see, so I show that here:

$$\mathbf{ijk} = \mathrm{i}\sigma_1 \mathrm{i}\sigma_2 [-\mathrm{i}\sigma_3] = -\mathrm{i}^3 \sigma_1\sigma_2\sigma_3 = -\mathrm{i}^4 \sigma_1\sigma_1 = -\sigma_1^2 = -1.$$

So I am pretty confident that my new generators $\mathbf{i}$ and $\mathbf{j}$ yield a group which is isomorphic to Q_4.

The multiplication table comes here:

$$\boldsymbol{T} = \begin{pmatrix} 1 & \mathbf{i} & \mathbf{j} & \mathbf{k} & -1 & -\mathbf{i} & -\mathbf{j} & -\mathbf{k} \\ \mathbf{i} & -1 & \mathbf{k} & -\mathbf{j} & -\mathbf{i} & 1 & -\mathbf{k} & \mathbf{j} \\ \mathbf{j} & -\mathbf{k} & -1 & \mathbf{i} & -\mathbf{j} & \mathbf{k} & 1 & -\mathbf{i} \\ \mathbf{k} & \mathbf{j} & -\mathbf{i} & -1 & -\mathbf{k} & -\mathbf{j} & \mathbf{i} & 1 \\ -1 & -\mathbf{i} & -\mathbf{j} & -\mathbf{k} & 1 & \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\mathbf{i} & 1 & -\mathbf{k} & \mathbf{j} & \mathbf{i} & -1 & \mathbf{k} & -\mathbf{j} \\ -\mathbf{j} & \mathbf{k} & 1 & -\mathbf{i} & \mathbf{j} & -\mathbf{k} & -1 & \mathbf{i} \\ -\mathbf{k} & -\mathbf{j} & \mathbf{i} & 1 & \mathbf{k} & \mathbf{j} & -\mathbf{i} & -1 \end{pmatrix}.$$

This could also be written in terms of the σ_i:

$$\boldsymbol{T} = \begin{pmatrix} e & \mathrm{i}\sigma_1 & \mathrm{i}\sigma_2 & -\mathrm{i}\sigma_3 & -e & -\mathrm{i}\sigma_1 & -\mathrm{i}\sigma_2 & \mathrm{i}\sigma_3 \\ \mathrm{i}\sigma_1 & -e & -\mathrm{i}\sigma_3 & -\mathrm{i}\sigma_2 & -\mathrm{i}\sigma_1 & e & \mathrm{i}\sigma_3 & \mathrm{i}\sigma_2 \\ \mathrm{i}\sigma_2 & \mathrm{i}\sigma_3 & -e & \mathrm{i}\sigma_1 & -\mathrm{i}\sigma_2 & -\mathrm{i}\sigma_3 & e & -\mathrm{i}\sigma_1 \\ -\mathrm{i}\sigma_3 & \mathrm{i}\sigma_2 & -\mathrm{i}\sigma_1 & -e & \mathrm{i}\sigma_3 & -\mathrm{i}\sigma_2 & \mathrm{i}\sigma_1 & e \\ -e & -\mathrm{i}\sigma_1 & -\mathrm{i}\sigma_2 & \mathrm{i}\sigma_3 & e & \mathrm{i}\sigma_1 & \mathrm{i}\sigma_2 & -\mathrm{i}\sigma_3 \\ -\mathrm{i}\sigma_1 & e & \mathrm{i}\sigma_3 & \mathrm{i}\sigma_2 & \mathrm{i}\sigma_1 & -e & -\mathrm{i}\sigma_3 & -\mathrm{i}\sigma_2 \\ -\mathrm{i}\sigma_2 & -\mathrm{i}\sigma_3 & e & -\mathrm{i}\sigma_1 & \mathrm{i}\sigma_2 & \mathrm{i}\sigma_3 & -1 & \mathrm{i}\sigma_1 \\ \mathrm{i}\sigma_3 & -\mathrm{i}\sigma_2 & \mathrm{i}\sigma_1 & e & -\mathrm{i}\sigma_3 & \mathrm{i}\sigma_2 & -\mathrm{i}\sigma_1 & -e \end{pmatrix}.$$

Writing this out using only the generators seems like a really bad idea:

$$\boldsymbol{T} = \begin{pmatrix} [\mathrm{i}\sigma_1]^4 & \mathrm{i}\sigma_1 & \mathrm{i}\sigma_2 & \mathrm{i}\sigma_1\mathrm{i}\sigma_1 & [\mathrm{i}\sigma_1]^2 & [\mathrm{i}\sigma_1]^2\mathrm{i}\sigma_1 & [\mathrm{i}\sigma_1]^2\mathrm{i}\sigma_2 & \mathrm{i}\sigma_2\mathrm{i}\sigma_1 \\ \mathrm{i}\sigma_1 & [\mathrm{i}\sigma_1]^2 & \mathrm{i}\sigma_1\mathrm{i}\sigma_1 & [\mathrm{i}\sigma_1]^2\mathrm{i}\sigma_2 & [\mathrm{i}\sigma_1]^2\mathrm{i}\sigma_1 & [\mathrm{i}\sigma_1]^4 & \mathrm{i}\sigma_2\mathrm{i}\sigma_1 & \mathrm{i}\sigma_2 \\ \mathrm{i}\sigma_2 & \mathrm{i}\sigma_2\mathrm{i}\sigma_1 & [\mathrm{i}\sigma_1]^2 & \mathrm{i}\sigma_1 & [\mathrm{i}\sigma_1]^2\mathrm{i}\sigma_2 & \mathrm{i}\sigma_1\mathrm{i}\sigma_1 & [\mathrm{i}\sigma_1]^4 & [\mathrm{i}\sigma_1]^2\mathrm{i}\sigma_1 \\ \mathrm{i}\sigma_1\mathrm{i}\sigma_1 & \mathrm{i}\sigma_2 & [\mathrm{i}\sigma_1]^2\mathrm{i}\sigma_1 & [\mathrm{i}\sigma_1]^2 & \mathrm{i}\sigma_2\mathrm{i}\sigma_1 & [\mathrm{i}\sigma_1]^2\mathrm{i}\sigma_2 & \mathrm{i}\sigma_1 & [\mathrm{i}\sigma_1]^4 \\ [\mathrm{i}\sigma_1]^2 & [\mathrm{i}\sigma_1]^2\mathrm{i}\sigma_1 & [\mathrm{i}\sigma_1]^2\mathrm{i}\sigma_2 & \mathrm{i}\sigma_2\mathrm{i}\sigma_1 & [\mathrm{i}\sigma_1]^4 & \mathrm{i}\sigma_1 & \mathrm{i}\sigma_2 & \mathrm{i}\sigma_1\mathrm{i}\sigma_1 \\ [\mathrm{i}\sigma_1]^2\mathrm{i}\sigma_1 & [\mathrm{i}\sigma_1]^4 & \mathrm{i}\sigma_2\mathrm{i}\sigma_1 & \mathrm{i}\sigma_2 & \mathrm{i}\sigma_1 & [\mathrm{i}\sigma_1]^2 & \mathrm{i}\sigma_1\mathrm{i}\sigma_1 & [\mathrm{i}\sigma_1]^2\mathrm{i}\sigma_2 \\ [\mathrm{i}\sigma_1]^2\mathrm{i}\sigma_2 & \mathrm{i}\sigma_1\mathrm{i}\sigma_1 & [\mathrm{i}\sigma_1]^4 & [\mathrm{i}\sigma_1]^2\mathrm{i}\sigma_1 & \mathrm{i}\sigma_2 & \mathrm{i}\sigma_2\mathrm{i}\sigma_1 & [\mathrm{i}\sigma_1]^2 1 & \mathrm{i}\sigma_1 \\ \mathrm{i}\sigma_2\mathrm{i}\sigma_1 & [\mathrm{i}\sigma_1]^2\mathrm{i}\sigma_2 & \mathrm{i}\sigma_1 & [\mathrm{i}\sigma_1]^4 & \mathrm{i}\sigma_1\mathrm{i}\sigma_1 & \mathrm{i}\sigma_2 & [\mathrm{i}\sigma_1]^2\mathrm{i}\sigma_1 & [\mathrm{i}\sigma_1]^2 \end{pmatrix}.$$

I will just stick to the first one.

Problem statement

Identify the order of the group Q_4.

The multiplication table is an 8×8 matrix, so the order of this group is 8.

Problem statement

Identify the order of each of its elements.

We have

$$1^m = 1, \quad \mathbf{i}^4 = 1, \quad \mathbf{j}^4 = 1, \quad \mathbf{k}^4 = 1, \quad [-1]^2 = 1, \quad [-\mathbf{i}]^4 = 1, \quad [-\mathbf{j}]^4 = 1, \quad [-\mathbf{k}]^4 = 1,$$

where the orders can be read off the needed exponent to get the elements to 1.

1.2 Conjugacy classes

Problem statement

What are the conjugacy classes of this group?

I again did this with the method that I have used on the previous problem set and got:

$$[1] = \{1, -1\}$$
$$[\mathbf{i}] = \{\mathbf{i}, -\mathbf{i}\}$$
$$[\mathbf{j}] = \{\mathbf{j}, -\mathbf{j}\}$$
$$[\mathbf{k}] = \{\mathbf{k}, -\mathbf{k}\}$$

1.3 Proper invariant subgroups

Problem statement

What are the proper invariant subgroups of this group?

Those proper invariant subgroups have to be unisons of conjugacy classes. The number of elements in those unions must be a divisor the order of the whole group, in this case eight (Lagrange's theorem).

The invariant groups that I found are:

$$\{1, -1\}$$
$$\{1, -1, \mathbf{i}, -\mathbf{i}\}$$
$$\{1, -1, \mathbf{j}, -\mathbf{j}\}$$
$$\{1, -1, \mathbf{k}, -\mathbf{k}\}$$

Other unions are now allowed since they would either violate Lagrange's theorem or be a trivial subgroup.

1.4 Quotient groups

Problem statement

What are the pertinent quotient groups?

The first one is

$$Q_4/\{1,-1\} = \{\{1,-1\},\{\mathbf{i},-\mathbf{i}\},\{\mathbf{j},-\mathbf{j}\},\{\mathbf{k},-\mathbf{k}\}\}\,.$$

Then for each of the other subgroups, they are constructed like this:

$$Q_4/\{1,-1,\mathbf{i},-\mathbf{i}\} = \{\{1,-1,\mathbf{i},-\mathbf{i}\},\{\mathbf{j},\mathbf{k},-\mathbf{j},-\mathbf{k}\}\}$$
$$Q_4/\{1,-1,\mathbf{j},-\mathbf{j}\} = \{\{1,-1,\mathbf{j},-\mathbf{j}\},\{\mathbf{i},\mathbf{k},-\mathbf{i},-\mathbf{k}\}\}$$
$$Q_4/\{1,-1,\mathbf{k},-\mathbf{k}\} = \{\{1,-1,\mathbf{k},-\mathbf{k}\},\{\mathbf{i},\mathbf{j},-\mathbf{i},-\mathbf{j}\}\}$$

1.5 Not isomorphic

Problem statement

Show that Q_4 is not isomorphic to D_4. Construct a generator representation.

Given is

$$D_4 = \mathrm{gp}\{c = \mathrm{i}\sigma_3, b = \sigma_1 : c^4 = e, b^2 = e, bcb^{-1} = c^{-1}\}.$$

We have to fill in the blanks in the representation for Q_4. That is:

$$Q_4 = \mathrm{gp}\{a = \mathrm{i}\sigma_1 = \mathbf{i}, b = \mathrm{i}\sigma_1 = \mathbf{j} : a^4 = e, b^2 = -1, bcb^{-1} = -\mathbf{jij} = -\mathbf{i} = -a = a^{-1}\}.$$

This is not exactly the same, the groups are probably not isomorphic to each other.

References

Penrose, Roger (2005). *Road to Reality*. 1. New York: Alfred A. Knopf. ISBN: 0-679-45443-8.