Disclaimer

This is a problem set (as turned in) for the module physics751.

This problem set is not reviewed by a tutor. This is just what I have turned in.

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[disclaimer]

physics751 – Group Theory

Problem Set 2

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problem number	achieved points	possible points
H 2.1		
total		

1 The Alternating Group $\mathscr{A}_N$

1.1 Subgroup

Problem Statement

Prove that the set of all even permutations forms a subgroup of the symmetric group $\mathscr{S}_N$.

As always, there are the four group axioms to show.

Closure Given a permutation P, it can be broken down into cycles. The number of elements in the ith cycle is n_i. So (123) has $n = 3$. The order of the whole permutation was given in the lecture as

$$W = \sum_i [n_1 - 1].$$

When another permutation is added, the cycles are written as a juxtaposition. The order will be the sum of the respective orders of the permutations. The sum of two even numbers is again even, so the resulting permutation is another even permutation.

Unity The cycle (1) is even and the neutral element.

Inverse For any cycle, the inverse can be formed with elements reversed. The inverse of multiple cycle is the reverse order of inverse cycles. The order of the permutation does not change in this way. Therefore, an inverse can always be found among the even permutations.

Associativity This subgroup in spe takes it juxtaposition operation $\circ$ from the symmetric group which was associative with any permutations. This property is not list when only a subset of the permutations is taken into account.

Since all axioms are fulfilled, this subset forms a group itself. I think that it even is a normal subgroup, since $\bar{P}\mathscr{S}_N = \mathscr{S}_N\bar{P}$, which is also somewhat used in the next part of this problem.

1.2 Order

Problem Statement

Derive its order.

Any even permutations directly turns into an odd one when the cycle (12) is applied to it. Any odd permutation turns into an even one when the same cycle is applied. Applying it twice gives unity: $(12)^2 = e$.

I think that $(12) \circ \mathscr{A}_N$ will give a set of only odd permutations. Those do not form a group because the closure cannot be met. The combination of two odd permutation gives an even permutation. However, when (12) is applied to that set again, it must be $\mathscr{A}_N$ again, since only the unity e was applied to it. By association law,

$$[(12)(12)]\mathscr{A}_N = (12)(12)\mathscr{A}_N,$$

where the multiplication is read in the usual right-to-left way on the right side. Since the order of $\mathscr{A}_N$ must not change when applying e to it (rearrangement theorem), the order of $(12)\mathscr{A}_N$ must the same as $\mathscr{A}_N$ itself. That would be $N!/2$.