Group Theory II

Part (a)

$$\hat{P}_{kk}^{\Gamma_{IR}} |F\rangle = \frac{l_{IR}}{h} \sum_R D^{\Gamma_{IR}}(R)^*_{kk} \hat{P}_R |F\rangle$$

$$= \underbrace{\frac{l_{IR}}{h} \sum_R D^{\Gamma_{IR}}(R)^*_{kk} \hat{P}_R \sum_{\Gamma_{IR}} \sum_m \alpha_m^{\Gamma_{IR}} |\Gamma_{IR}, m\rangle}_{\overset{!}{=} \sum_m \delta_{mk}}$$

Show that $\overset{!}{=} \sum_m \delta_{mk}$.

$$= \frac{l_{IR}}{h} \sum_R \underbrace{D^{\Gamma_{IR}}(R)^*_{kk}}_{\langle \Gamma_{IR}, k | \hat{P}_R | \Gamma_{IR}, k\rangle} \hat{P}_R \sum_{\Gamma_{IR}} \sum_m \alpha_m^{\Gamma_{IR}} |\Gamma_{IR}, m\rangle$$

That is only nonzero if $|\Gamma_{IR}, k\rangle$ is an eigenstate of $\hat{P}_R$. So $m = k$.

$$= \frac{l_{IR}}{h} \hat{P}_r \alpha_m^{\Gamma_{IR}} |\Gamma_{IR}, m\rangle$$

Project onto $\langle \Gamma_{IR}, m'|$

$$\langle \Gamma_{IR}, m' | \frac{l_{IR}}{h} \sum_R \underbrace{D^{\Gamma_{IR}}(R)^*_{kk}} \hat{P}_R \sum_{\Gamma_{IR}} \sum_m \alpha_m^{\Gamma_{IR}} |\Gamma_{IR}, m\rangle$$

$$= D^{\Gamma_{IR}}(R)_{m'm}$$

$$= \frac{l_{IR}}{h} \underbrace{\sum_R D^{\Gamma_{IR}}(R)^*_{kk} \, D^{\Gamma_{IR}'}(R)_{m'm}}_{\frac{h}{l_{IR}} \delta_{\Gamma_{IR}\Gamma_{IR}'} \delta_{km} \delta_{km'}} \sum_{\Gamma_{IR}} \sum_m \alpha_m^{\Gamma_{IR}}$$

$$= \alpha_M^{\Gamma_{IR}}$$

This could also be done by inserting a complete set of eigenstates

$$\sum_n \sum_m |\Gamma_n, m\rangle\langle\Gamma_n, m| = 1$$

and use that $\hat{P}$ does not change the irreducible representation.

Part (b)

Apply $\hat{P}$ once and one gets $\alpha_k |k\rangle$. Now when applying it again on this "arbitrary" vector, we project out $|k\rangle$ again with

Part (c)

$-1 \cdot [-1] = 1$. The adjoint $\dagger$ does not change the -1, so this seems easy.

Part (d)

Finite group $\rightarrow$ matrix representation
$\rightarrow$ get 2 matrices in $SU(2)$ $\rightarrow$ 2 element