A5.2

2014-11-10

Part 1

Let V be a d-dimensional vector space over $F = \mathbb{C}$ or $\mathbb{R}$, S non-singular $S \in GL(d, F)$, $\ker(S) = \{0\}$, $\operatorname{Im}(S) = V$. Let $\{v_1, \ldots, v_d\}$ be a basis of V and $S(v_i) = w_i$

$\Rightarrow \{w_1, \ldots, w_d\}$ are linear independent.

Let G be a group and D^G a representation.

$$D^G w_n = D^G S(v_i)$$

$$= D^G \sum_{m=1}^{d} S_{mn} v_m$$

$$= \sum_{m=1}^{d} S_{mn} D^G (v_m)$$

$$= \sum_{m=1}^{d} S_{mn} \sum_{k=1}^{d} D^G_{km} v_k$$

$$\left[D^G v_m\right]_j = D_{jk} v_m^k$$

$$= \sum_{m,k=1}^{d} S_{mn} D^G_{km} \sum_{l=1}^{d} S^{-1}_{lk} w_l$$

$$= \sum_{l,k,m=1}^{d} S^{-1}_{lk} D^G_{km} S_{mn} w_l$$

$$= \sum_{l}^{d} D^{G'}_{ln} w_l$$

thus $\{w_1, \dots, w_n\}$ is a set of basis vectors for the equivalent representation

$$D^{G'} = S^{-1} D^G S$$

Since $w_i = S(v_i) = \sum_m S_{mi} v_m$, the effect of a similarity trafo is the rearrangement of the basis of the space without changing the space itself. $\square$

Part 2

Let S be unitary D^G unitary rep of G and $D^{G'} = S^{-1} D^G S$ the equivalent representation.

Note that $\langle S v_i, S v_j \rangle = \langle v_i, v_j \rangle = \langle D^G v_i, D^G v_j \rangle$ as S and D^G are unitary.

Then orthonormality is preserved since:

$$\begin{aligned}
\langle v_i, v_j \rangle &= \langle S v_i, S v_2 \rangle \\
&= \langle D^G S v_i, D^G S v_j \rangle \\
&= \langle S^{-1} D^G S v_i, S^{-1} D^G S, v_i \rangle \\
&= \langle D^{G'} v_i, D^{G'} v_j \rangle
\end{aligned}$$

$D^{G'} = S^{-1} D^G S$ is unitary since

$$\begin{aligned}
\langle D^{G'} a, D^{G'} b \rangle &= \sum_{i,j=1}^{d} a_i^* b_j \langle D^{G'} v_i, D^{G'} v_j \rangle \\
&= \sum_{i,j=1}^{d} a_i^* b_j \langle v_i, v_j \rangle \\
&= (a, b) \quad \text{for } \forall a, b \in V
\end{aligned}$$