A 3.1

2014-10-27

Part 1

$g^m = e$

$\{e, g^1, g^2, \dots, g^{m-1}\}$ is subgroop $\cong C_m$

Theorem of Lagrange: m is divisor of N_G.

Part 2

Theorem of Lagrange $\Rightarrow$ G cannot have non-trivial subgroups.

- $p \geq 2$: $\exists g \neq e$
- closed: $m < p \Rightarrow g^m \in G$
- $\forall$ positive $m < p \Rightarrow g^m \neq e$
- all g^m with $1 \leq m < p$ must be diffent (else one of the g^m would be unity)
- $g^p = e \Rightarrow$ group must be C_p

Part 3

- Subgroups of all abelian groups are normal since $ghg^{-1} = h$ $\forall h \in H \subset G$ and $\forall g \in G$.
- The only abelian groups without proper subgroups are C_p with p a prime.

Part 4

$H \subset G$ is a normal subgroup.

Then $\forall h \in H \ \forall g \in G \ \exists h' \in H: \ ghg^{-1} = h'$

$\Rightarrow \quad gHg^{-1} \subseteq H$

show: $\quad H \subseteq gHg^{-1} \qquad \forall \bar{h} \in H \ \exists h \in H, \ g \in G: \ \bar{h} = ghg^{-1}$

Proof.
From the definition of a normal subgroup:

$$g^{-1}\bar{h}g = h \in H.$$

$$\Rightarrow \quad \underset{\in H}{\bar{h}} = \underset{\in gHg^{-1}}{ghg^{-1}}$$

$\square$

$$\Rightarrow \quad gHg = H \quad \Leftrightarrow \quad gH = Hg$$

Part 5

- if H is not normal $\Rightarrow$ $\exists \ g \in G$ with $g \notin H$ and one $h \in H \Rightarrow ghg^{-1} \notin H$.

let us show that this induces

$$(gH)(g^{-1}H) \neq gg^{-1}H$$

Let's denote $g_h = ghg^{-1} \notin H$

$$\Rightarrow (gH)(g^{-1}H) = (gHg^{-1})H \supseteq g_h H \neq H$$

since $\exists h \in H$ such that $ghg^{-1} =: g_h \notin H$.

Thus $(gH)(g^{-1}H) \neq H \equiv eH \equiv gg^{-1}H$

Thus the above defined product is inconsistent if $H \subset G$ is not invariant.