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This problem set is not reviewed by a tutor. This is just what I have turned in.

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[disclaimer]

physics7501 – Advanced Quantum Field Theory

Problem Set 7

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Problem	Achieved points	Possible points
You can't say "functional" without "fun"		15
Total		15

This document consists of 5 pages.

1 You can't say "functional" without "fun"

1.1 Interaction

In general we have

$$W(J) = \frac{1}{2} \int \mathrm{d}^4 x \, \mathrm{d}^4 y \, J(x) D_\mathrm{F}(x - y) J(y) \,.$$

The $J(x)$ consists of two summands, therefore $W(J)$ consists of four summands. We will focus on one off-diagonal summand here.

$$W_{12}(J) = \frac{1}{2} \int \mathrm{d}^4 x \, \mathrm{d}^4 y \, \delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}_1) D_\mathrm{F}(x - y) \delta^{(3)}(\boldsymbol{y} - \boldsymbol{x}_2) \,.$$

The spatial integrations will fix $\boldsymbol{x}$ and $\boldsymbol{y}$, x^0 and y^0 are still free and have to be integrated over. Splitting up the temporal and spatial arguments in the propagator, we have

$$= \frac{1}{2} \int \mathrm{d}x^0 \, \mathrm{d}y^0 \, D_\mathrm{F}(x^0 - y^0, \boldsymbol{x} - \boldsymbol{y}) \,.$$

Then we can insert the scalar propagator and obtain

$$= \frac{1}{2} \int \mathrm{d}x^0 \, \mathrm{d}y^0 \frac{\mathrm{d}^4 p}{[2\pi]^4} \frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\epsilon} \exp(-\mathrm{i}p^0[x^0 - y^0]) \exp(\mathrm{i}\boldsymbol{p} \cdot [\boldsymbol{x} - \boldsymbol{y}]) \,.$$

The time integration probably is in an interval of length T. So we write the time integration to the end and make the limits explicit.

$$= \frac{1}{2} \frac{\mathrm{d}^4 p}{[2\pi]^4} \frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\epsilon} \exp(\mathrm{i}\boldsymbol{p} \cdot [\boldsymbol{x} - \boldsymbol{y}]) \int_t^{t+T} \mathrm{d}x^0 \int_t^{t+T} \mathrm{d}y^0 \exp(-\mathrm{i}p^0[x^0 - y^0])$$

This is not handy, so we introduce new variables, $a := x^0 - y^0$ and $b := x^0 + y^0$. The Jacobi matrix of the inverse transformation is

$$\frac{\partial(x, y)}{\partial(a, b)} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} .$$

The determinant of this matrix is $1/2$ as common factors of a matrix enter the determinant to the power n where n^2 is the number of matrix elements. The change of variables in the integration is a pullback, therefore the Jacobian of the inverse transformation has to be used. The new integral therefore is

$$= \frac{1}{4} \frac{\mathrm{d}^4 p}{[2\pi]^4} \frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\epsilon} \exp(\mathrm{i}\boldsymbol{p} \cdot [\boldsymbol{x} - \boldsymbol{y}]) \int \mathrm{d}a \int \mathrm{d}b \exp(-\mathrm{i}p^0 a) \,.$$

Before thinking about the new integration domain too hard, we note that the integration over a looks like it could give a $\delta(-p^0)$ term if the integration was not bounded. Then a change in variables would give another minus sign (which we need) and set the remaining p^0 in the denominator to zero. The remaining $\int \mathrm{d}b$ would have to give a factor of $4T$ to give the desired result. Since the integration over a is bounded, there is no real $\delta(-p^0)$ emerging from that. The integration over p^0 cannot be used to give $\delta(-a)$ as there are other factors of p^0 in the expression which alter the oscillatory behavior.

1.2 Yukawa-potential

Missing

1.3 Degrees of polarization

We start off with the well known fact that

$$-\eta_{\mu\nu} = -\eta_{\mu\nu} \,.$$

Then we can write the left side a tad more complicated by introducing more dummy indices. The idea is that only the diagonal elements of the metric tensor of special relativity are nonzero. Therefore we can write all those four nonzero elements explicitly as

$$-\delta^0_\mu \delta^0_\nu + \sum_a \delta^a_\mu \delta^a_\nu = -\eta_{\mu\nu} \,.$$

The Latin index a runs over the numbers in the set $\{1,2,3\}$. Next we can move the first summand over to the other side. Our bootstrapped relation then is

$$\sum_a \delta^a_\mu \delta^a_\nu = -\eta_{\mu\nu} + \delta^0_\mu \delta^0_\nu .$$

On the left side we introduce the polarization basis vectors $\epsilon^a_\mu = \delta^a_\mu$. On the right side we write the Kronecker symbols at components of $\tilde{\boldsymbol{k}} = (m,0,0,0)$ which is the momentum four-vector of a particle at rest. To do so, we have to divide by m^2 in order to remove the factor of m that $\tilde{\boldsymbol{k}}$ introduces. This procedure then takes us to

$$\sum_a \epsilon^a_\mu \epsilon^a_\nu = -\eta_{\mu\nu} + \frac{\tilde{k}_\mu \tilde{k}_\nu}{m^2} .$$

This is still in the rest frame of the particle in question. We need to boost it with a boost matrix $\boldsymbol{\Lambda}$. All the index are lowered, so actually one would need the inverse transformation. Since one could just choose the rapidity with opposite sign we will not really bother with all the inverses here. Either way, we now boost everything to momentum $\boldsymbol{k}$.

$$\Lambda_\mu{}^\alpha \Lambda^\nu{}_\beta \sum_a \epsilon^a_\alpha \epsilon^a_\beta = -\Lambda_\mu{}^\alpha \Lambda_\nu{}^\beta \eta_{\alpha\beta} + \frac{\Lambda_\mu{}^\alpha \Lambda_\nu{}^\beta \tilde{k}_\alpha \tilde{k}_\beta}{m^2} .$$

We now define all those explicit boosts away. The polarization vectors are to be taken at the momentum $\boldsymbol{k}$. The vector $\tilde{\boldsymbol{k}}$ becomes $\boldsymbol{k}$ with the transformation. The metric tensor does not change under the Lorentz transformation. Since that is the defining property of a Lorentz transformation it is rather pointless to show that explicitly. We talked about the Penrose diagrammatic notation last week, so just for the sake of including it, we have done so. The diagram is shown in Figure 2 with a short explanation of the notation in Figure 1. To get back to the formulas, we now have

$$\sum_a \epsilon^a_\mu(\boldsymbol{k}) \epsilon^a_\nu(\boldsymbol{k}) = -\eta_{\mu\nu} + \frac{k_\mu k_\nu}{m^2} .$$

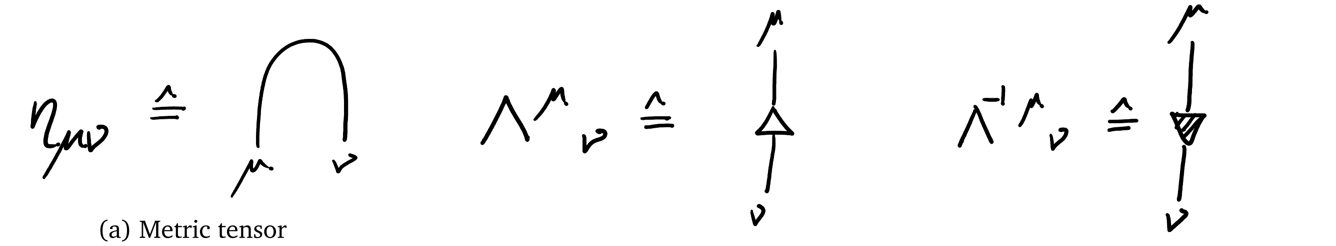
Now one can factor out a minus sign and arrive at the desired equation:

$$\sum_a \epsilon^a_\mu(\boldsymbol{k}) \epsilon^a_\nu(\boldsymbol{k}) = -\left[\eta_{\mu\nu} - \frac{k_\mu k_\nu}{m^2} \right] .$$

Writing it down in this way really makes clear where this extra $k_\mu k_\nu$ comes from. The sum over a should really include $a = 0$, the temporal polarization basis vector, as well. However, this is not a physical polarization state, it needs to be subtracted. Also one can clearly see here that this does not work as neatly with massless particles as there is no rest frame for massless particles.

1.4 Attraction of like charges

Missing



(a) Metric tensor

(b) Orthogonal transformation

(c) Inverse transformation

Figure 1: Elements for the Penrose diagrammatic notation used in Figure 2.

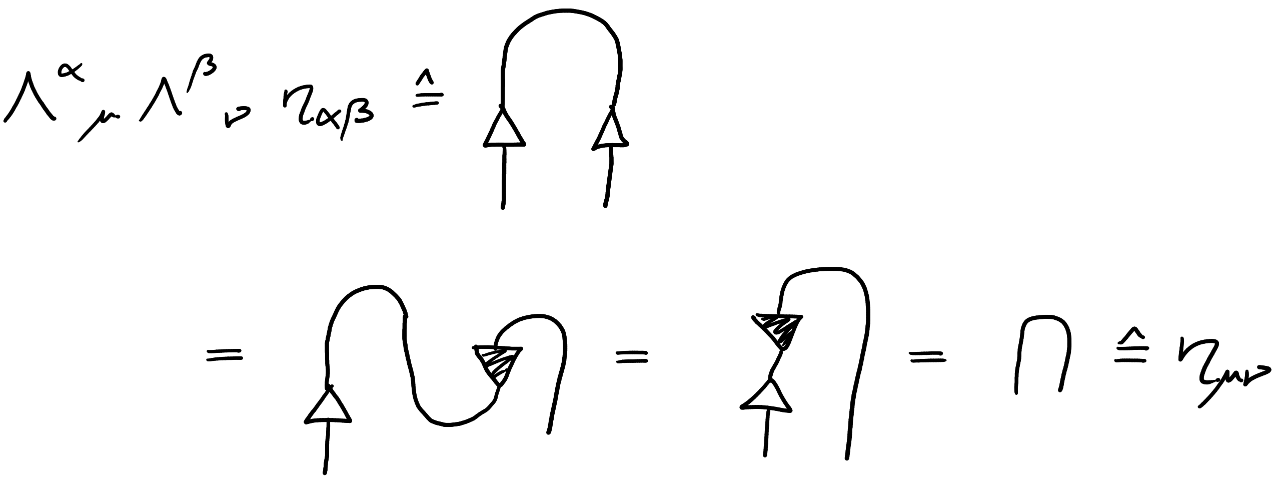


Figure 2: Showing that the metric tensor is not altered by a Lorentz transformation. In the first diagrammatic equality we have used that the transpose of the inverse is the matrix itself. Then we "wiggled" the paths to remove the two extra metric tensors in between to a Kronecker symbol. The matrix and the inverse are removed to give a Kronecker symbol as well. We end up with the metric tensor.

1.5 Massive gravity

A 4×4 tensor has 16 degrees of freedom. This means that the space GL(4) has 16 generators or basis vectors. Their number is limited by a couple constraints:

- First is the symmetry of the tensor. This removes six degrees of freedom, there are only ten left.
- The requirement that the basis tensors must not have any trace removes another degree of freedom.
- There is also some identity which looks like the Ward identity. A on-shell momentum four-vectors has three degrees of freedom. It would seem likely that the four equations (indexed by ν) remove three degrees of freedom. The on-shell equation is another constraint as well, so all in all it removes four degrees of freedom.

The normalization of the basis vectors does not remove any degree of freedom in the sense of a tensor space considered above. It only fixes the basis to some particular value.

All in all we have $16-6-1-4=5$. So the number of basis elements matches the number that we want to have.

The whole identity that we have to show can be boosted to the rest frame as all the terms are $[4,0]$-tensors and transform in the same way. Therefore it suffices to look at the equation in the rest frame. There the tensor $\boldsymbol{G}$ takes the form

$$G_{\mu\nu} = \eta_{\mu\nu} - \frac{k_\mu k_\nu}{m^2} = \eta_{\mu\nu} - \delta^0_\mu \delta^0_\nu \simeq \mathrm{diag}(0,-1,-1,-1)_{\mu\nu}\,.$$

1.6 Energy stress tensor

So here we do not have $\boldsymbol{G} = 8\pi G \boldsymbol{T}$ where the first $\boldsymbol{G}$ is the Einstein tensor and the second G is the gravitational constant? Does the $\boldsymbol{G}$ from the problem here has any relation to the Einstein tensor (except that it is symmetric)?