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physics7501 – Advanced Quantum Field Theory

Problem Set 6

Martin Ueding

mu@martin-ueding.de

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Tutor: Thorsten Schimannek

Problem	Achieved points	Possible points
Grassmann numbers and the fermionic path integral		15
Total		15

This document consists of 8 pages.

1 Grassmann numbers and the fermionic path integral

1.1 Taylor series and shift

The Grassmann numbers themselves are the generators of the algebra. It probably makes sense to add 1 to the generators also in order to have $\mathbf{C}$ be part of the algebra as well. We will call that set $\mathbf{G}$.[1] The function f then is

$$f : \mathbf{G} \to \mathbf{G} .$$

Since the Grassmann numbers anticommute they square to zero.[2] This also means that a function of a single Grassmann variable cannot have arbitrary form. Analytic functions can always be written as a power

[1]The notation $\mathbb{C}$ and $\mathbb{G}$ might be more common for fields. I like to go with the ISO 80000-2 standard (which allows both variants) and also take the heritage of those letters (like $\mathbb{C}$) into account. In LaTeX they are called "blackboard bold" which means this way of writing letters is the attempt to write bold on the blackboard. Since we have more powerful typesetting in LaTeX than on the board, I will sure make use of them and display the letters in bold. Same goes for underline for emphasis, which is a no-go in typed documents. Perhaps it is not the best idea to deviate too much from common notation even though its usage in a powerful typesetting system feels rather backwards.

[2]On the Wikipedia article about Grassmann numbers they are called "non-zero square roots of zero" which I find a quite intriguing thought. Almost ten years ago the concept of $\sqrt{-1}$ has baffled me, now this is the next step. This time I know that field axioms are not god-given and therefore algebra structures can be defined like one wishes to furnish those axioms. Still, it is an interesting way of stating the squaring to zero feature.

series, functions can at most be linear in Grassmann variables which really limits their complexity. Any function can be written as

$$f(\theta) = A + B\theta\,,$$

where we use the same notation as Peskin and Schroeder (1995, p. 299). Higher order terms in θ vanish directly.

Taylor series The Taylor series of such a function in θ just has two terms as higher powers of θ will vanish. Also higher derivatives in θ will also vanish when one looks at the general form motivated above.

Integral The integration must be invariant under shifts in the integration variable. We have

$$\int \mathrm{d}\theta\, f(\theta) = \int \mathrm{d}\theta\, [A + B\theta]$$

as the general form. Now shifting θ by η does not change $\mathrm{d}\theta$ and we have

$$= \int \mathrm{d}\theta\, [A + B[\theta + \eta]]$$

which we can expand to yield

$$= \int \mathrm{d}\theta\, [A + B\theta + B\eta]\,.$$

It is best to regroup the terms such that the constant and linear terms clearly pop out:

$$= \int \mathrm{d}\theta\, [\underbrace{A + B\eta}_{A'} + B\theta]\,.$$

The result of the integration must not be changed when $A \to A'$ it performed. Therefore the integral must not depend on A as it is a linear function and has to be rather simple. The only dependence can be B then. A definition that fulfils this is

$$\int \mathrm{d}\theta\, [A + B\theta] = kB\,.$$

Could this k be a Grassmann number, i.e. $k \in \mathbf{G}$? B is definitely a regular complex number, so the product would not directly vanish. However, the integral would then be a $\mathbf{G}$-linear function which does not play well with integrations in $\mathbf{C}$. It therefore makes sense to define $k \in \mathbf{C}$ and therefore $kB = c$ is a regular complex number.

Integration and Differentiation On the problem set it is noted that integration and differentiation are equivalent. Since there are only two possible monoms of Grassmann variables (A and $B\theta$), this is easy

Side question

U(1) only has one generator, 1. Therefore the group $\mathrm{SO}(1) = \{1\}$ is a quite trivial group. The groups $\mathrm{SO}(n)$ all have $n^2 - 1$ generators and $\mathrm{U}(n)$ should have n^2 generators. Is it in general that

$$\mathrm{U}(n) \simeq \mathrm{U}(1) \times \mathrm{SO}(n)$$

holds?

to check. The integral and derivative with respect to θ of the first term is zero. The integral and derivative of the second is just B. Setting $k = 1$ makes this truly the same thing which is probably very handy along the road.

1.2 Unitary transformation

The unitary transformation is probably a *special* unitary transformation. In the case of a general unitary transformation the determinant might be a complex phase factor which does change the integrals unless complex conjugated counter-terms are involved as well. So U is taken from $\mathrm{SO}(n)$ and $\mathrm{U}(1)$ is excluded here? Since the problem talks about matrices, $\mathrm{SO}(n)$ would not really be a problem. So this just applies when there are multiple fields? Peskin and Schroeder (1995, p. 301) show that the integral with $\mathrm{d}\theta^* \, \mathrm{d}\theta$ is invariant under $\mathrm{U}(n)$ but integrals with $\mathrm{d}\theta$ depend on $\det(U)$ which might have a phase factor.

We will now show that the transformation introduces a factor on $\det(U)$. The invariance has to be evaluated with respect of that determinant then which is unity for the special transformations.

First we show a useful identity. We are given

$$\prod_a \theta_i = \frac{1}{n!} \epsilon^{ij\dots l} \theta_i \theta_j \dots \theta_l \,.$$

We have used a on the left side in order to distinguish from i. We will need that distinction in a bit. Now we want to “divide” by the Levi-Civita symbol ϵ. We do that in the following way: If we have an equation that has free indices $ij\dots l$ and we contract it with the present Levi-Civita symbol, we have found the above equation “divided by ϵ” such that it is on the other side. We take ϵ to be normalized to $n!$. Then the desired relation is

$$\epsilon_{ij\dots l} \prod_a \theta_a = \theta_i \theta_j \dots \theta_l \,.$$

Multiplication and contraction with ϵ will give a factor of $n!$ on the left side. Moving that to the right gives the $1/n!$ needed there.

Now we can rewrite the transformed product. This is

$$\prod_a \theta'_a = \frac{1}{n!} \epsilon^{ij\dots l} \theta'_i \theta'_j \dots \theta'_l \,.$$

Then we can expand the θ' in terms of θ. We obtain

$$= \frac{1}{n!} \epsilon^{ij\dots l} U_i^{i'} U_j^{j'} \dots U_l^{l'} \theta_{i'} \theta_{j'} \dots \theta_{l'} \,.$$

Now we can use the relation shown above. This will transform the θ on the right side to

$$= \frac{1}{n!} \epsilon^{ij\dots l} U_i^{i'} U_j^{j'} \dots U_l^{l'} \epsilon_{i'j'\dots l'} \prod_a \theta_a \,.$$

The fancy factor in front of the product symbol is just the determinant of the matrix $\boldsymbol{U}$. This mean that we have

$$= \det(\boldsymbol{U}) \prod_a \theta_a \,.$$

The change with an arbitrary linear transformation therefore changes the product by its determinant.’

At this point the similarity to the differential forms which are also antisymmetric is very apparent. A linear transformation introduces the *Jacobian* which is nothing else than this determinant of the linear transformation matrix.

Again, if $\boldsymbol{U} \in \mathrm{SO}(n)$, then $\det(\boldsymbol{U}) = 1$ and the integral is invariant under that transformation.

1.3 Integral identities

We have seen before that the integral “stamps out” the factor in front of the Grassmann variable in question. This will simplify the first integral significantly. A change of variables with an unitary transformation does not change the integral. The matrices of interest to quantum theory can be diagonalized by an unitary transformation. Therefore we can rewrite the given integral[3] as

$$\left[\int \prod_a \mathrm{d}\bar{\theta}_a \, \mathrm{d}\theta_a\right] \exp(-\bar{\theta}_i B_{ij} \theta_j) = \left[\int \prod_a \mathrm{d}\bar{\theta}'_a \, \mathrm{d}\theta'_a\right] \exp\left(-\sum_i \bar{\theta}'_i b_i \theta'_i\right),$$

where we name the eigenvalues of $\boldsymbol{B}$ just b_i. Now the integrals are actually independent. This only works because pairs of Grassmann numbers commute with other pairs. We can rewrite the integrals as

$$= \prod_i \int \mathrm{d}\bar{\theta}'_i \, \mathrm{d}\theta'_i \exp\left(-\bar{\theta}'_i b_i \theta'_i\right).$$

Due to the antisymmetry of the variables we can write out all the summands of the exponential fully. That is a first, writing out a normal exponential is quite time consuming ☺. Anyway, we then have

$$= \prod_i \int \mathrm{d}\bar{\theta}'_i \, \mathrm{d}\theta'_i \left[1 - \bar{\theta}'_i \theta'_i b_i\right].$$

The integration stamps out the b_i and the remainder is

$$= \prod_i b_i$$

which is just

$$= \det(\boldsymbol{B}).$$

The other one can be shown using the differentiation again. So we obtain the desired bilinear using a partial integration. The bilinear is in the wrong order in the exponential. Luckily we obtain another minus sign from the exponential. Anticommuting the two Grassmann variables will remove the minus sign and leave us with a nice expression without any signs.

$$\left[\int \prod_a \mathrm{d}\bar{\theta}_a \, \mathrm{d}\theta_a\right] \theta_k \bar{\theta}_l \exp(-\bar{\theta}_i B_{ij} \theta_j) = \left[\int \prod_a \mathrm{d}\bar{\theta}_a \, \mathrm{d}\theta_a\right] \frac{\partial}{\partial B_{kl}} \exp(-\bar{\theta}_i B_{ij} \theta_j)$$

Assuming all behaves well we can pull out the derivative.

$$= \frac{\partial}{\partial B_{kl}} \left[\int \prod_a \mathrm{d}\bar{\theta}_a \, \mathrm{d}\theta_a\right] \exp(-\bar{\theta}_i B_{ij} \theta_j)$$

This integral was already computed and we insert the result.

$$= \frac{\partial}{\partial B_{kl}} \det(\boldsymbol{B})$$

[3] The exterior derivate was missing in the volume elements. The integral would then be over a 0-form which I do not really know what that would be. I guess it would be zero as the result would be a −1-form which is not possible. Anyway, I just added the “d” as that is clear from the context ☺.

The determinant can be written as a lot of Bs and Levi-Civita symbols.

$$= \frac{\partial}{\partial B_{kl}} \frac{1}{n!} \epsilon_{ab\dots d} B^a{}_{a'} B^b{}_{b'} \dots B^d{}_{d'} \epsilon^{a'b'\dots d'}$$

The product rule (or Leibniz rule if we want to sound fancy) will create n terms. They are all equal in the following sense: The matrix element differentiated will have an index pair of i and i'. This has the same position in both Levi-Civita symbols. We can permute the indices on both symbols at the same time such that i and i' are the front element. Then we rename all the other dummy indices such that the terms are the same. Our combined terms are then

$$= \frac{n}{n!} \epsilon_{kb\dots d} B^b{}_{b'} \dots B^d{}_{d'} \epsilon^{lb'\dots d'} .$$

According to Penrose (2005, Fig. 13.7) the inverse of a matrix is very similar to that, except that we also need to divide by the $n!$ times determinant to get it right. Since we do not have that term here, we have to multiply with it.

$$= \det(\boldsymbol{B})(\boldsymbol{B}^{-1})_{kl}$$

That is the final result then.

1.4 Generating functional

The generating functional was introduced to allow the build-up of correlation functions using functional derivatives. We had a homework where we worked with those kind of integrals where a source term J was added. At the time I did a functional derivative with respect to the matrix elements. You wrote that we are supposed to do that with respect to J. I do see why that is the case now.

This functional was not really derived, it was rather defined and shown that it is quite useful. The same probably happens here. We propose the generating function for the free Dirac theory to include some source terms. The correlation functions can have ψ and $\bar\psi$ terms in them, therefore we need two source terms. Those source terms are taken to be η and $\bar\eta$. This has the nice side effect that the Lagrangian density will stay real for every choice of complex η and ψ.

Having that in mind, it is not a big leap to define[4]

$$Z(\bar\eta, \eta) = \int \mathscr{D}\bar\psi \, \mathscr{D}\bar\psi \, \exp\left(\mathrm{i} \int \mathrm{d}^4x \left[\bar\psi[\mathrm{i}\partial\!\!\!/ - m]\psi + \bar\eta\psi + \bar\psi\eta \right] \right) .$$

From here we have to show that it takes the form in the second line which relies on this being a free theory where all the Gaussian integrals are only quadratic in the exponent.

The integration by parts as done by Peskin and Schroeder (1995, p. 290) is not needed here, we have a first order differential operator $\hat O$ here such that we have a bilinear of the form $\bar\psi\hat O\psi$ already. Therefore we can directly strive to complete the "square". Here it rather means to complete the terms linear in the

[4] I do not like to write $Z[\bar\eta, \eta]$ as that notation looks like Z being multiplied with the commutator of the two variables. Also I want to be able to distinguish $H(t - t_0)$ from $H[t - t_0]$, i.e. the Hamiltonian *evaluated* at $t - t_0$ and the Hamiltonian *multiplied* at $t - t_0$, both is a reasonable thing. I chose to represent this with different parentheses. The whole rationale is at `http://martin-ueding.de/en/physics/function-notation/index.html` I you want to know more about my idiosyncrasies ☺.

spinors to bilinears. The following substitutions seem to work rather well:

$$\psi'(x) = \psi(x) - \mathrm{i} \int \mathrm{d}^4 y\, S_\mathrm{F}(x-y)\eta(y), \qquad \bar{\psi}'(x) = \bar{\psi}(x) - \mathrm{i} \int \mathrm{d}^4 z\, S_\mathrm{F}(x-z)\bar{\eta}(y).$$

The second is just the hermitian conjugate of the other one. Then we insert that into the Lagrangian density. Writing out the whole thing here would just kill the paper. The density is

$$\mathscr{L} = \bar{\psi}[\mathrm{i}\not{\partial} - m]\psi + \bar{\eta}\psi + \bar{\psi}\eta.$$

With the new fields it is

$$\begin{aligned} &= \left[\bar{\psi}'(x) + \mathrm{i} \int \mathrm{d}^4 z\, S_\mathrm{F}(x-z)\bar{\eta}(z)\right][\mathrm{i}\not{\partial} - m]\left[\psi'(x) + \mathrm{i} \int \mathrm{d}^4 y\, S_\mathrm{F}(x-y)\eta(y)\right] \\ &\quad + \bar{\eta}\left[\psi'(x) + \mathrm{i} \int \mathrm{d}^4 y\, S_\mathrm{F}(x-y)\eta(y)\right] + \left[\bar{\psi}'(x) + \mathrm{i} \int \mathrm{d}^4 z\, S_\mathrm{F}(x-z)\bar{\eta}(z)\right]\eta. \end{aligned}$$

This has to be factored out. Some of the terms will be old Lagrangian density with the primed spinors. We write this as $\mathscr{L}'$. The new terms are now

$$\begin{aligned} &= \mathscr{L}' - \int \mathrm{d}^4 z\, S_\mathrm{F}(x-z)\bar{\eta}(z)[\mathrm{i}\not{\partial} - m] \int \mathrm{d}^4 y\, S_\mathrm{F}(x-y)\eta(y) \\ &\quad + \mathrm{i}\bar{\psi}'(x)[\mathrm{i}\not{\partial} - m] \int \mathrm{d}^4 y\, S_\mathrm{F}(x-y)\eta(y) + \mathrm{i} \int \mathrm{d}^4 z\, S_\mathrm{F}(x-z)\bar{\eta}(z)[\mathrm{i}\not{\partial} - m]\psi'(x) \\ &\quad + \mathrm{i}\bar{\eta} \int \mathrm{d}^4 y\, S_\mathrm{F}(x-y)\eta(y) + \mathrm{i} \int \mathrm{d}^4 z\, S_\mathrm{F}(x-z)\bar{\eta}(z)\eta. \end{aligned}$$

In the book they stress that the propagator is the Green's function to the kinetic operator. Therefore it is the "operator inverse" that will give a Dirac δ-distribution. That is then integrated over. Perhaps it gives an imaginary unit or some other phase factor. Either way, the first three summands simplify this way. We add the explicit function argument to the remaining factors, this is just x by default.

$$\begin{aligned} &= \mathscr{L}' - \int \mathrm{d}^4 y\, \bar{\eta}(x) S_\mathrm{F}(x-y)\eta(y) + \mathrm{i}\bar{\psi}'(x)\eta(x) + \mathrm{i}\bar{\eta}(x)\psi'(x) \\ &\quad + \mathrm{i} \int \mathrm{d}^4 y\, \bar{\eta}(x) S_\mathrm{F}(x-y)\eta(y) + \mathrm{i} \int \mathrm{d}^4 z\, \bar{\eta}(z) S_\mathrm{F}(x-z)\eta(x). \end{aligned}$$

The last two terms are equal to the first one except for the sign. It would be very handy if we had a phase factor from the Green's function property that we have just used. Then two of the three terms would cancel and one would be left with just a factor of one imaginary unit. Also the $\bar{\psi}'\eta$ and the hermitian conjugate term would cancel with the source terms present in $\mathscr{L}'$. Then the final thing would be

$$\mathscr{L} = \mathscr{L}_0 + +\mathrm{i} \int \mathrm{d}^4 z\, \bar{\eta}(z) S_\mathrm{F}(x-z)\eta(x).$$

Plugging this into the exponential function where we have $\exp(-\mathrm{i} \int \mathrm{d}^4 x \mathscr{L})$, we would get the desired Z_0 times the exponential with a negative sign in it.

I am sorry I cannot look closer into that right now. Please tell me in the tutorial (or on this sheet) what the phase factor is if I plug in the Green's function into the equation itself.

References

Penrose, Roger (2005). *Road to Reality*. 1. New York: Alfred A. Knopf. ISBN: 0-679-45443-8.

Peskin, Michael E. and Daniel V. Schroeder (1995). *An Introduction to Quantum Field Theory*. Westview Press. ISBN: 978-0-201-50397-5.