Disclaimer

This is a problem set (as turned in) for the module physics615.

This problem set is not reviewed by a tutor. This is just what I have turned in.

All problem sets for this module can be found at
http://martin-ueding.de/de/university/msc_physics/physics615/.

[disclaimer]

TPP Sheet 12 Martin Ueding

1. U(1) Higgs mechanism

Now we use the U(1) symmetry in the choice of the phase of the vev.

Part (a)

The potential is given as

$$V = m^2 |\varphi|^2 + \frac{\lambda}{2} |\varphi|^4.$$

$$V = m^2 \varphi^* \varphi + \frac{\lambda}{2} (\varphi^* \varphi)^2.$$

$$\varphi_1 := \varphi \qquad \varphi_2 := \varphi^*$$

$$V = m^2 \varphi_1 \varphi_2 + \frac{\lambda}{2} \varphi_1^2 \varphi_2^2$$

$$\frac{\partial^2 V}{\partial z_i \partial z_j} = \frac{\partial^2 V}{\partial \varphi_k \partial \varphi_\ell} \frac{\partial \varphi_k}{\partial z_i} \frac{\partial \varphi_\ell}{\partial z_j}$$

$$\frac{\partial V}{\partial \varphi} = m^2 \varphi^* + \lambda \varphi^{*2} \varphi$$

$$\frac{\partial^2 V}{\partial \varphi \partial \varphi} = \lambda \varphi^{*2} \qquad \frac{\partial^2 V}{\partial \varphi^* \partial \varphi} = m^2 + 2\lambda \varphi^* \varphi$$

$$\frac{\partial^2 V}{\partial \varphi_\ell \varphi_k} = \begin{pmatrix} \lambda \varphi^{*2} & m^2 + 2\lambda \varphi^* \varphi \\ m^2 + 2\lambda \varphi^* \varphi & \lambda \varphi^2 \end{pmatrix}_{\ell k}$$

$$\frac{\partial \varphi}{\partial \eta} = \frac{1}{\sqrt{2}} \qquad \frac{\partial \varphi^*}{\partial \eta} = \frac{1}{\sqrt{2}} \qquad \frac{\partial \varphi}{\partial \xi} = \frac{i}{\sqrt{2}} \qquad \frac{\partial \varphi^*}{\partial \xi} = -\frac{i}{\sqrt{2}}.$$

$$\frac{\partial \varphi_i}{\partial z_j} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}_{ij}$$

Construct chain rule carefully.

$$\mathcal{M}^i{}_j = \frac{\partial^2 V}{\partial z_i \, \partial z^j} = \underbrace{\frac{\partial^2 V}{\partial \varphi_a \, \partial \varphi^b}}_{M'^a{}_b} \underbrace{\frac{\partial \varphi_a}{\partial z_i}}_{J^i{}_a} \underbrace{\frac{\partial \varphi^b}{\partial z^j}}_{J^b{}_j}$$

So $\mathcal{M}^i{}_j = J^i{}_a \, M'^a{}_b \, J^b{}_j$

$$\frac{\partial \varphi_a}{\partial z_i} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}^i{}_a \Rightarrow J^b{}_j = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}^b{}_j$$

The mass matrix therefore is:

$$\mathcal{M} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} \lambda \varphi^{*2} & m^2 + 2\lambda \varphi^* \varphi \\ m^2 + 2\lambda \varphi^* \varphi & \lambda \varphi^2 \end{pmatrix} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$$

Compute with Mathematica

$$= \frac{1}{2} \begin{pmatrix} 2m^2 + \lambda(\varphi^2 + 4\varphi^*\varphi + \varphi^2) & -i\lambda(\varphi - \varphi^*)(\varphi + \varphi^*) \\ -i\lambda(\varphi - \varphi^*)(\varphi + \varphi^*) & 2m^2 - \lambda(\varphi^2 - 4\varphi^*\varphi + \varphi^{*2}) \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 2m^2 + \lambda(\varphi^2 + 4\varphi^*\varphi + \varphi^2) & -i\lambda(\varphi^2 - \varphi^{*2}) \\ -i\lambda(\varphi^2 - \varphi^{*2}) & 2m^2 - \lambda(\varphi^2 - 4\varphi^*\varphi + \varphi^{*2}) \end{pmatrix}$$

And now replace remaining φ and φ^* with vev.

$$\frac{1}{2} \begin{pmatrix} 2m^2 + \lambda(v^2 e^{2i\beta} + 4v^2 + v^2 e^{-2i\beta}) & -i\lambda(v^2 e^{2i\beta} - v^2 e^{-2i\beta}) \\ -i\lambda(v^2 e^{2i\beta} - v^2 e^{-2i\beta}) & 2m^2 - \lambda(v^2 e^{2i\beta} - 4v^2 + v^2 e^{-2i\beta}) \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 2m^2 + \lambda v^2(4 + 2\cos(2\beta)) & 2\lambda v^2 \sin(2\beta) \\ 2\lambda v^2 \sin(2\beta) & 2m^2 - \lambda v^2(4 - 2\cos(2\beta)) \end{pmatrix}$$

Part (b)

Extract eigensystem with Mathematica.

$$\Lambda_1 = m^2 + \lambda v^2 \qquad \vec{V}_1 = \begin{pmatrix} \cot(2\beta) - \csc(2\beta) \\ 1 \end{pmatrix}$$

$$\Lambda_2 = m^2 + 3\lambda v^2 \qquad \vec{V}_2 = \begin{pmatrix} \cot(2\beta) + \csc(2\beta) \\ 1 \end{pmatrix}$$

In the limit $\beta \to 0$ as in class these eigenvectors are not sensible. One has to take the limit first and then find the eigenvectors. Those are then

$$\vec{V}_1^{(0)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \vec{V}_2^{(0)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The eigenvectors for $\beta \neq 0$ are not normalized. We use

no factor 2

$$\cot(2\beta) - \csc(2\beta) = -\tan(\beta)$$

$$\cot(2\beta) + \csc(2\beta) = \cot(\beta)$$

$$\vec{V}_1 = \begin{pmatrix} \cot(2\beta) - \csc(2\beta) \\ 1 \end{pmatrix} = \begin{pmatrix} -\tan(\beta) \\ 1 \end{pmatrix} \propto \begin{pmatrix} -\sin(\beta) \\ \cos(\beta) \end{pmatrix}$$

$$V_2 = \begin{pmatrix} \cot(2\beta) + \csc(2\beta) \\ 1 \end{pmatrix} = \begin{pmatrix} \cot(\beta) \\ 1 \end{pmatrix} \propto \begin{pmatrix} \cos(\beta) \\ \sin(\beta) \end{pmatrix}$$

So we have $\gamma = \beta$.

In general we have a matrix M, eigenvector matrix V and eigenvalue matrix Λ.
We have:

$$M v_i = \lambda_i v_i$$

This needs to hold in the transformed basis as well. We have

$$V e_i = v_i \iff V^{-1} v_i = e_i$$

Therefore it must be

$$M = V \Lambda V^{-1} \iff V^{-1} M V = \Lambda$$

We have
$$V = \begin{pmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{pmatrix}$$
here. The O on the set is defined with

$$\mathcal{M}^2 O^T = O^T \operatorname{diag}(m_1^2, m_2^2).$$

So $O \mathcal{M}^2 O^T$ is diagonal. That means that $O = V^T$ and $\beta = \gamma$.

Also note that the minimum occurs at

$$2 m^2 |\varphi| + 2\lambda |\varphi|^3 = 0$$

$$|\varphi| \left(m^2 + \lambda |\varphi|^2 \right) = 0$$

$$|\varphi|^2 = -\frac{m^2}{\lambda} = v^2$$

$$m^2 + \lambda v^2 \rightsquigarrow m^2 - \lambda \frac{m^2}{\lambda} = 0$$ Goldstone boson

$$m^2 + 3\lambda v^2 \rightsquigarrow m^2 - 3\lambda \frac{m^2}{\lambda} = -2m^2$$ Higgs field

2 Gauge invariance in the Abelian Higgs model

Part (a)

We need a term with $A_\mu A^\mu$ in it for the mass. The second term in Equation (7) gives us

$$i(-i)g^2 A_\mu A_\mu \varphi^* \varphi = g^2 A^2 |\varphi|^2$$

Inserting the vev and writing

$$\frac{1}{2} 2 g^2 v^2 A^2$$

gives $m_A = \sqrt{2}\, gv$.

Part (b)

The desired vertex is caused by some $AA\varphi$ term in the Lagrangian. We need to find that.

The field φ consists of a constant and the two fields η and ξ. We have in the term

$$(D^\mu \varphi)^\dagger (D_\mu \varphi)$$

one term with two A fields: $g^2 A^2 |\varphi|^2$

We do not insert the vev here, yet. We expand $|\varphi|^2$.

$$g^2 A^2 \left(v e^{-i\beta} + \frac{\eta - i\xi}{\sqrt{2}}\right)\left(v e^{i\beta} + \frac{\eta + i\xi}{\sqrt{2}}\right)$$

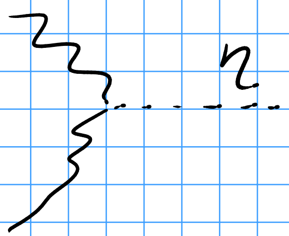
$$= g^2 A^2 \left[v^2 + v e^{-i\beta} \frac{\eta + i\xi}{\sqrt{2}} + v e^{i\beta} \frac{\eta - i\xi}{\sqrt{2}} + \frac{\eta^2 + \xi^2}{2} \right]$$

Focus on those terms:

$$g^2 A^2 v \frac{1}{\sqrt{2}} \left[\eta \underbrace{\left(e^{-i\beta} + e^{i\beta}\right)}_{2\cos(\beta)} + i\xi \underbrace{\left(e^{-i\beta} - e^{i\beta}\right)}_{-2i\sin(\beta)} \right]$$

$$= g A^2 v \sqrt{2} \left[\eta \cos(\beta) + \xi \sin(\beta) \right]$$

This gives an



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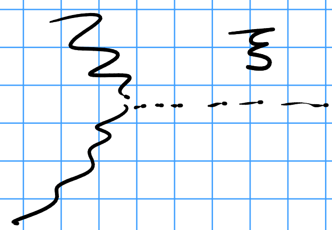

vertex.

I am not really sure how the ver is supposed to fit in here. In class there was $\beta = 0$ and the second vertex does not occur. Is that it?

Part (c)

Equation (3) is

$$\varphi(x) = v e^{i\beta} + \frac{1}{\sqrt{2}}\left[\eta(x) + i\,\xi(x)\right]$$

The eigenvectors in η/ξ-space are the physical fields:

$$G(x) = -\sin(\beta)\,\eta(x) + \cos(\beta)\,\xi(x)$$

$$h(x) = \cos(\beta)\,\eta(x) + \sin(\beta)\,\xi(x)$$

Here we want to go into the other direction. So

$$\begin{pmatrix} \eta \\ \xi \end{pmatrix} = \begin{pmatrix} \cos & \sin \\ -\sin & \cos \end{pmatrix} \begin{pmatrix} G \\ h \end{pmatrix}$$

$$\eta(x) = \cos(\beta)\,G(x) + \sin(\beta)\,h(x)$$

$$\xi(x) = -\sin(\beta)\,G(x) + \cos(\beta)\,h(x)$$

Inserting that into (3) gives

$$\varphi(x) = v e^{i\beta} + \frac{1}{\sqrt{2}}\Big[\cos(\beta)\,G(x) + \sin(\beta)\,h(x) - i\sin(\beta)\,G(x) + i\cos(\beta)\,h(x)\Big]$$

$$= v e^{i\beta} + \frac{1}{\sqrt{2}}\Big[G(x)\left[\cos(\beta) - i\sin(\beta)\right] \underset{-i}{+} h(x)\left[\underset{+}{i}\cos(\beta) \underset{-i}{+} \sin(\beta)\right]\Big]$$

$$= v e^{i\beta} + \frac{1}{\sqrt{2}}\left[G(x)\,e^{-i\beta} - i\,h(x)\,e^{-i\beta}\right]$$

$$= v e^{i\beta} + \frac{1}{\sqrt{2}}\,e^{-i\beta}\left[G(x) - i\,h(x)\right]$$