Disclaimer

This is a problem set (as turned in) for the module physics615.

This problem set is not reviewed by a tutor. This is just what I have turned in.

All problem sets for this module can be found at
http://martin-ueding.de/de/university/msc_physics/physics615/.

[disclaimer]

1. Brute-force computation in SU(3)

I do not see a point in doing those computations by hand. So I let *Mathematica* do the job for me.

(a) The SU(2) subalgebra

First we need to define the first three generators.

```
In[24]:= t1 := 1 / 2 {{0, 1, 0}, {1, 0, 0}, {0, 0, 0}}

In[25]:= t2 := 1 / 2 {{0, -I, 0}, {I, 0, 0}, {0, 0, 0}}

In[26]:= t3 := 1 / 2 {{1, 0, 0}, {0, -1, 0}, {0, 0, 0}}

In[27]:= t4 := 1 / 2 {{0, 0, 1}, {0, 0, 0}, {1, 0, 0}}

In[28]:= t5 := 1 / 2 {{0, 0, -I}, {0, 0, 0}, {I, 0, 0}}

In[29]:= t6 := 1 / 2 {{0, 0, 0}, {0, 0, 1}, {0, 1, 0}}

In[30]:= t7 := 1 / 2 {{0, 0, 0}, {0, 0, -I}, {0, I, 0}}

In[31]:= t8 := 1 / (2 Sqrt[3]) {{1, 0, 0}, {0, 1, 0}, {0, 0, -2}}
```

For later convenience we define a list of the eight generators:

```
In[37]:= t := {t1, t2, t3, t4, t5, t6, t7, t8}
```

Written as ordinary matrices, they take the usual form:

```
In[38]:= Map[MatrixForm, t]
```

Out[38]=
$$\left\{\begin{pmatrix}0&\frac{1}{2}&0\\\frac{1}{2}&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&-\frac{i}{2}&0\\\frac{i}{2}&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}\frac{1}{2}&0&0\\0&-\frac{1}{2}&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&\frac{1}{2}\\0&0&0\\\frac{1}{2}&0&0\end{pmatrix},\right.$$
$$\left.\begin{pmatrix}0&0&-\frac{i}{2}\\0&0&0\\\frac{i}{2}&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&0&\frac{1}{2}\\0&\frac{1}{2}&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&0&-\frac{i}{2}\\0&\frac{i}{2}&0\end{pmatrix},\begin{pmatrix}\frac{1}{2\sqrt{3}}&0&0\\0&\frac{1}{2\sqrt{3}}&0\\0&0&-\frac{1}{\sqrt{3}}\end{pmatrix}\right\}$$

For instance the commutator [T_1, T_2] must be i T_3, and in fact it is:

```
In[33]:= t1.t2 - t2.t1 == I t3

Out[33]= True
```

One could go on and check other terms, but it is clear that the generators T_1 to T_3 are just the Pauli matrices with additional padding. That makes it SU(2) only, albeit in a three dimensional reducible representation.

(b) Orthogonality relation

The following matrix contains all the Tr(T_i, T_j) terms. As you can see, only the diagonal is occupied.

In[40]:=
```
MatrixForm[Table[Tr[t[[i]].t[[j]]], {i, 1, 8}, {j, 1, 8}]]
```

Out[40]//MatrixForm=

$$\begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

The factor C that we are looking for is just 1/2.

(c) Quadratic Casimir operator

We just compute that sum of the squared generators directly:

In[43]:=
```
MatrixForm[Sum[t[[a]].t[[a]], {a, 1, 8}]]
```

Out[43]//MatrixForm=

$$\begin{pmatrix} \frac{4}{3} & 0 & 0 \\ 0 & \frac{4}{3} & 0 \\ 0 & 0 & \frac{4}{3} \end{pmatrix}$$

We see that the factor is 4/3.

It works out nicely. The dimension of this (defining) representation is 3. The dimension of the adjoint is 8 as there are eight generators. Now we have

In[44]:=
```
4 / 3 * 3 == 8 * 1 / 2
```

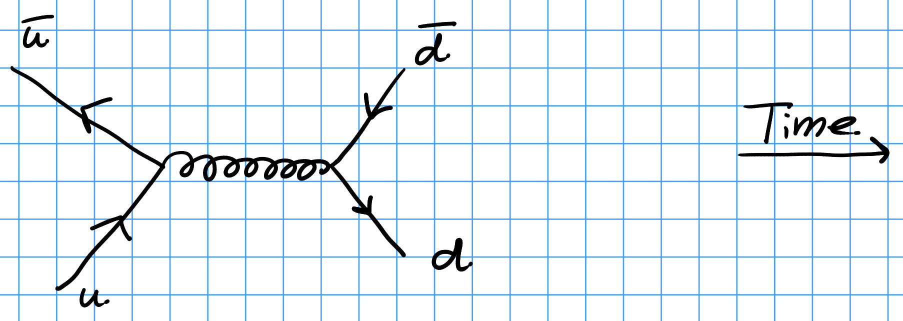
Out[44]=
```
True
```

which is true. So the identity holds at least for the three-dimensional representation.

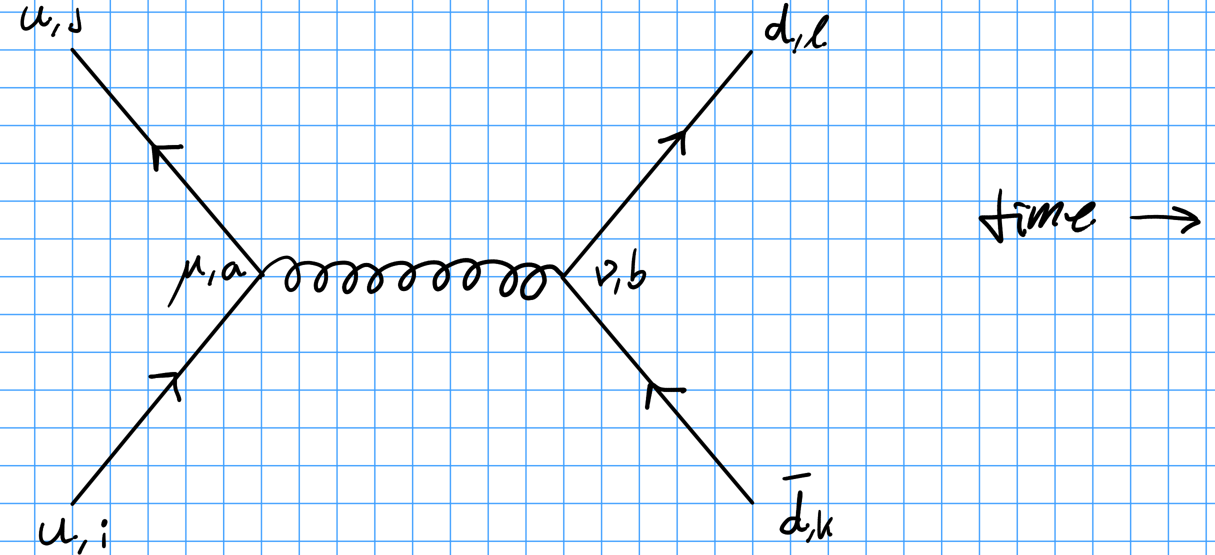
2. Quark – Antiquark annihilation

Part (a)

Interaction with gluon:



We need some indices, namely Lorentz, color and gluon indices.



So we have the following rules

- Fermion – gluon vertex: $\quad i g \gamma^\mu T^a{}_{ij}$
  (Dirac indices are supressed)

- Gluon propagator:

$$-i\,\delta_{ab}\left[\frac{g_{\mu\nu}}{k^2+i\varepsilon} - [1-\xi]\,\frac{k_\mu k_\nu}{(k^2)^2}\right]$$

The incoming an outgoing fermions have u and $\bar{u}$ respectively. Now we can assemble the diagram.

$$i\mathcal{M} =$$

$$-\bar{u}_j \, i g \gamma^\mu T^a_{ji} u_i \, i \delta_{ab} \left[\frac{g_{\mu\nu}}{k^2 + i\varepsilon} - [1-\xi] \frac{k_\mu k_\nu}{(k^2)^2} \right] \bar{d}_\ell \, i g \gamma^\nu T^b_{\ell k} d_k$$

Now that's a nice matrix element compared to QED. We have five types of indices:

- Lorentz: $\mu, \nu, \ldots$ $\quad 0, \ldots, 3$
- Dirac (suppressed): $\alpha, \beta, \ldots$ $\quad 1, \ldots, 4$
- Color: $i, j, \ldots$ $\quad 1, 2, 3$
- Gluon: $a, b, \ldots$ $\quad 1, \ldots, 8$
- Spin (suppressed): $s, t, \ldots$ $\quad 1, 2$

The sum over the gluon colors is already built-in with the contraction of a and b. We can simplify the matrix element to give

$$i g^2 \bar{u}_j \gamma^\mu T^a_{ji} u_i \left[\frac{g_{\mu\nu}}{k^2 + i\varepsilon} - [1-\xi] \frac{k_\mu k_\nu}{(k^2)^2} \right] \bar{d}_\ell \gamma^\nu T^a_{\ell k} d_k .$$

The color generators commute with everything else, all indices are explicit. We have

$$T^a_{ji} T^a_{\ell k} .$$

For these sums he have

$$(\lambda^a)_i^{\ j} (\lambda^a)_k^{\ \ell} = 2 \left[\delta_i^{\ell} \delta_k^{j} - \frac{1}{3} \delta_i^{j} \delta_k^{\ell} \right].$$

With $T^a = \frac{1}{2} \lambda^a$ we can use that relation here.

$$T^a_{ji} T^a_{\ell k} = \frac{1}{4} \lambda^a_{ji} \lambda^a_{\ell k} = \frac{1}{2} \delta^\ell_i \delta^j_k - \frac{1}{12} \delta^j_i \delta^\ell_k .$$

We should also average over spins in a physical setup. But that is not important here.

Anyway, I'm not sure how to proceed here. Is the color factor the $T^a_{ji} T^a_{\ell k}$ that appears in the invariant matrix element?

Part (b)

From the wording here I presume that the color factor is $T^a_{ji} T^a_{\ell k}$. The modulus squared is that times its hermitian conjugate. In the physicist's convention we have hermitian generators, therefore this is:

$$T^a_{ji} T^a_{\ell k} T^b_{k\ell} T^b_{ij} \qquad \text{(summation only over a and b)}$$

Is there more that I can do here?

Part (c)

Summing over k and ℓ now.

$$\sum_{a,b} \sum_{k,\ell} T^a_{ji} \underbrace{T^a_{\ell k} T^b_{k\ell}}_{Tr(T^a T^b) = \frac{1}{2}\delta^{ab}} T^b_{ij}$$

$$= \frac{1}{2} \sum_a T^a_{ji} T^a_{ij}$$

Part (d)

Well, this is a waste of time to do by hand. The $\frac{1}{9}\sum_{i,j=1}^{3}$ averages over incoming colors.

2. Quark-Antiquark annihilation

(d) Explicit averaging

First we need to define the first three generators.

In[2]:=
```
t1 := 1 / 2 {{0, 1, 0}, {1, 0, 0}, {0, 0, 0}}
```

In[3]:=
```
t2 := 1 / 2 {{0, -I, 0}, {I, 0, 0}, {0, 0, 0}}
```

In[4]:=
```
t3 := 1 / 2 {{1, 0, 0}, {0, -1, 0}, {0, 0, 0}}
```

In[5]:=
```
t4 := 1 / 2 {{0, 0, 1}, {0, 0, 0}, {1, 0, 0}}
```

In[6]:=
```
t5 := 1 / 2 {{0, 0, -I}, {0, 0, 0}, {I, 0, 0}}
```

In[7]:=
```
t6 := 1 / 2 {{0, 0, 0}, {0, 0, 1}, {0, 1, 0}}
```

In[8]:=
```
t7 := 1 / 2 {{0, 0, 0}, {0, 0, -I}, {0, I, 0}}
```

In[9]:=
```
t8 := 1 / (2 Sqrt[3]) {{1, 0, 0}, {0, 1, 0}, {0, 0, -2}}
```

For later convenience we define a list of the eight generators:

In[10]:=
```
t := {t1, t2, t3, t4, t5, t6, t7, t8}
```

The problem asks to look at all nine combinations. Then it makes it more specific by giving a summation over the indices i and j. That will make the whole result just a single element. This does not make too much sense as there are not nine combinations, then. We look at the sum over a with fixed i and fixed j. This gives 9 terms which are organized in a matrix indexed with i and j. The result is:

In[25]:=
```
r := Table[1 / 2 Sum[t[[a]][[i, j]] t[[a]][[j, i]], {a, 1, 8}], {i, 1, 3}, {j, 1, 3}]
```

In[24]:=
```
MatrixForm[r]
```

Out[24]//MatrixForm=

$$\begin{pmatrix} \frac{1}{6} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{6} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{6} \end{pmatrix}$$

The sum over all the elements here is just 2:

In[27]:=
```
Sum[r[[a, b]], {a, 1, 3}, {b, 1, 3}]
```

Out[27]= 2

We can also sum over i and j now. Basically we then have

In[18]:=
```
Sum[t[[a]][[i, j]] t[[a]][[j, i]], {a, 1, 8}, {i, 1, 3}, {j, 1, 3}] / 9
```

Out[18]= $\frac{4}{9}$

I am not really sure what this is supposed to tell me.