Disclaimer

This is a problem set (as turned in) for the module physics615.

This problem set is not reviewed by a tutor. This is just what I have turned in.

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http://martin-ueding.de/de/university/msc_physics/physics615/.

[disclaimer]

TPP Homework 7 or 9 Martin Ueding

So the filename says 7, the sheet says 9, so I'd take 7.

1 Non-Abelian gauge symmetry: Lagrangian

Part (a)

The definition of D_μ is wrong on the sheet. It contains ψ which does not make any sense. Actually it should be

$$D_\mu = \partial_\mu + ig A_{\mu a} T_a .$$

We want to have

$$(D_\mu \psi)' = U(x) D_\mu \psi$$

There we can expand the exponential map and obtain some generator.

$$\psi \to \psi' = U(x) \psi = [1 + ig \chi_a(x) T_a] \psi + O(\chi^2)$$

$$D_\mu \psi \to D_\mu \psi' = D_\mu U(x) \psi$$

Do we simply want $[D_\mu, U(x)] \psi = 0$?

I will try that by computing both terms individually. First the normal order:

$$D_\mu U(x)\psi = [\partial_\mu + ig A_{\mu a} T_a] U(x)\psi$$

$$= U_{,\mu}(x)\psi + U(x)\psi_{,\mu} + ig A_{\mu a} T_a U(x)\psi$$

Insert $U(x)$ and factor that out.

$$= U(x)[ig \chi_{b,\mu}(x) T_b \psi + \psi_{,\mu}] + ig A_{\mu a} T_a U(x)\psi$$

Now the other side.

$$U(x) D_\mu \psi = U(x)[\partial_\mu + ig A_{\mu a} T_a]\psi$$

The commutator then is:

$$U(x)\, ig\, \chi_{b,\mu}(x) T_b \psi + ig A_{\mu a} T_a U(x)\psi - U(x)\, ig A_{\mu a} T_a \psi$$

We can drop the ψ now. This shall be zero. Therefore we move $U(x)A$ to the other side.

$$U(x) A_{\mu a} T_a = U(x)\chi_{b,\mu}(x) T_b + A_{\mu a} T_a U(x)$$

$$[T_a, 1 + ig\chi_c(x) T_c] = ig\chi_c(x)[T_a, T_c] = -g f_{acd} \overset{\chi_c(x)}{T_d}$$

$$U(x) A_{\mu a} T_a = U(x)\chi_{b,\mu} T_b + A_{\mu a} U(x) T_a - g f_{acd} A_{\mu a} \chi_c(x) T_d$$

So from here we have after index exchange

$$A'_{\mu a} = A_{\mu a} + \chi_{a,\mu} - g f_{abc} A_{\mu c} \chi_b(x)$$

So those conditions are equivalent to first order.

Part (b)

That's easy. $D_\mu \psi$ transforms like ψ.

Then $\bar{\psi}\, i \not{D} \psi \longrightarrow \bar{\psi}\, \underbrace{U^\dagger(x)\, U(x)}_{\mathbb{1}}\, i \not{D} \psi$

Part (c)

Plug in and evaluate. We use the commutator to make it more compact.

$$[D_\mu, D_\nu] = [\partial_\mu + ig A_{\mu a} T_a, \partial_\nu + ig A_{\nu b} T_b]$$

$$= \underbrace{[\partial_\mu, \partial_\nu]}_{} + ig[\partial_\mu, A_{\nu b}] T_b$$

0 due to Schwartz' theorem.

$$+ ig[A_{\mu a}, \partial_\nu] T_a - g^2 A_{\mu a} A_{\nu b} [T_a, T_b]$$

$$= ig[\partial_\mu, A_{\nu b}] T_b - ig[\partial_\nu, A_{\mu a}] T_a - i g^2 A_{\mu a} A_{\nu b} f_{abc} T_c$$

$$\xrightarrow{\frac{1}{ig}} [\partial_\mu, A_{\nu b}] T_b - [\partial_\nu, A_{\mu a}] T_a - g A_{\mu a} A_{\nu b} f_{abc} T_c$$

$$\partial_\mu A_{\nu b} \psi - A_{\nu b} \partial_\mu \psi = A_{\nu b, \mu}$$

$$= A_{\nu a, \mu} T_a - A_{\mu a, \nu} T_a - g A_{\mu b} A_{\nu c} f_{abc} T_a$$

Extract T_a and voila:

$$F^a_{\mu\nu} = A_{\nu a, \mu} - A_{\mu a, \nu} - g A_{\mu b} A_{\nu c} f_{abc}.$$

Part (d)

$$\left[[D_\mu, D_\nu]\psi\right]' \text{ is } \left[ig\, F_{\mu\nu}\,\psi\right]'$$

And that is $ig\, F'_{\mu\nu}\,\psi' = ig\, F'_{\mu\nu}\,U(x)\,\psi$

On the other hand

$$U(x)\,[D_\mu, D_\nu]\,\psi = ig\; U(x)\, F_{\mu\nu}\,\psi$$

Set both sides equal:

$$ig\, F'_{\mu\nu}\, U(x)\,\psi = ig\; U(x)\, F_{\mu\nu}\,\psi$$

Cancel ig, cancel ψ, multiply from right with $U(x)$. Done.

For $O(\chi^a)$ transformation that is

$$F'_{\mu\nu} = \left[1 + ig\,\chi_a(x)\,T_a\right] F^b_{\mu\nu}\,T_b \left[1 - ig\,\chi_c(x)\,T_c\right]$$

(The T_a are hermitian matrices, χ is real).

$$F'_{\mu\nu} = F_{\mu\nu} + ig\,\chi_a(x)\,T_a\,F^b_{\mu\nu}\,T_b - ig\,F^b_{\mu\nu}\,T_b\,\chi_c(x)\,T_c$$
$$+ g^2\,\chi_a(x)\,T_a\,F^b_{\mu\nu}\,T_b\,\chi_c(x)\,T_c$$

$$= F_{\mu\nu} + ig\,\chi_a(x)\,F^b_{\mu\nu}\,[T_a, T_b]$$
$$+ g^2\,\chi_a(x)\,\chi_c(x)\,F^b_{\mu\nu}\,T_a\,T_b\,T_c$$

$$= F_{\mu\nu} - g f_{abc} \chi_b(x) F^c_{\mu\nu} T_a$$

$$+ g^2 \chi_a(x) \chi_c(x) F^b_{\mu\nu} T_a T_b T_c$$

That last summand has to vanish. It is symmetric in the indices a and c. So one can write the interesting terms as

$$\frac{1}{2} \chi_a \chi_c \left[T_a T_b T_c + T_c T_b T_a \right]$$

$$T_a T_b T_c = T_a T_c T_b + i f_{bcd} T_a T_d$$

$$= T_c T_a T_b + i f_{bcd} T_a T_d + i f_{acd} T_d T_b$$

$$= T_c T_b T_a + i f_{bcd} T_a T_d + i f_{acd} T_d T_b + i f_{abd} T_c T_d$$

Hmm

Part (e)

$$\mathrm{tr}\left(F_{\mu\nu}F^{\mu\nu}\right) \longrightarrow \mathrm{tr}\left(F_{\mu\nu}'F^{\mu\nu\prime}\right)$$

$$= \mathrm{tr}\left(U F_{\mu\nu} U^{-1} U F_{\mu\nu} U^{-1}\right) \underset{\text{cyclicity}}{=} \mathrm{tr}\left(F_{\mu\nu}F^{\mu\nu}\right)$$

Gauge invariant.

Part (f)

$$T^1 = \frac{1}{2}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \qquad T^2 = \frac{1}{2}\begin{pmatrix} & -i \\ i & \end{pmatrix} \qquad T^3 = \frac{1}{2}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

$$\sigma^i \sigma^j = \delta^{ij}\mathbb{1}_2 + \frac{1}{2}\varepsilon^{ijk}\sigma^k$$

$$\mathrm{tr}(\sigma^i\sigma^j) = \delta^{ij}\,\mathrm{tr}(\mathbb{1}_2) + \frac{1}{2}\varepsilon^{ijk}\,\mathrm{tr}(\sigma^k)$$

$$\delta^{ij}\cdot 2 \qquad\qquad \hookrightarrow 0$$

$$T^i = \frac{1}{2}\sigma^i$$

$$\mathrm{tr}(T^iT^j) = \frac{1}{4}\,\mathrm{tr}(\sigma^i\sigma^j) = \frac{1}{2}\delta^{ij}$$

Part (g)

$$\frac{1}{2}\,\mathrm{tr}_{color}\left(F_{\mu\nu}F^{\mu\nu}\right) = \frac{1}{2}F_{\mu\nu a}F^{\mu\nu a}\underbrace{\mathrm{tr}_{color}\left(T_a T^a\right)}_{\frac{1}{2}\delta^{aa} = \frac{1}{2}}$$

$$= \frac{1}{4}F_{\mu\nu a}F^{\mu\nu a}$$

2. Gauge (non-) invariance of hadronic wave functions

So what are those q_i things? They need to be from the SU(3) representation space. If the 3-dim representation is chosen, we have $q_i \in \mathbb{C}^3$. Then the q_i could be basis vectors in that space?

Part (a)

When I transform a trilinear, do I transform all of them at once?

$$\varepsilon_{ijk} \; q_i \; q_j \; q_k \to \varepsilon_{ijk} \; U_{ii'} \; U_{jj'} \; U_{kk'} \; q_{i'} \; q_{j'} \; q_{k'}$$

It does not hurt to insert another antisymmetrization and decouple that. The spinors anticommute anyway

$$\underbrace{\frac{1}{n!} \varepsilon_{ijk} \; U_{ii'} \; U_{jj'} \; U_{kk'} \; \varepsilon_{i'j'k'}}_{\det(U)=1} \; \underbrace{\varepsilon_{i''j''k''} \; q_{i''} \; q_{j''} \; q_{k''}}_{\text{Original wave function.}}$$

Part (b)

$$\bar{q}_i q_i \to \bar{q}_i \, U^{\dagger i}{}_j \, U^j{}_k \, q_k = \bar{q}_i \; \delta_{ik} \, q_k = \bar{q}_k q_k$$

Part (c)

$$\bar{q}_1 q_1 \to q_i \, U^{\dagger i}{}_1 \, U^1{}_k \, q_k$$

No clear inverse, not invariant