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[disclaimer]

physics606 – Advanced Quantum Theory

Problem Set 11

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problem number	achieved points	possible points
1		19
2		12
3		5
total		36

1 Two-particle scattering

The notation is a bit strange. The center of mass system has an asterisk in the superscript, the lab frame has an italic (although it should be roman according to ISO standard) "L" in it. We will drop this "L" and just have asterisk and non-asterisk variables.

1.1 Galilean transformation

In the center of mass frame, there is no total momentum:

$$\boldsymbol{p}_1^* + \boldsymbol{p}_2^* = \boldsymbol{0}.$$

The given velocities v have no specification whether they are before of after the scattering. To avoid any confusion, we define the ones before to be v, the after to be w. The transformation is a simple boost which is $\boldsymbol{v}_i^* = \boldsymbol{v}_i - \boldsymbol{B}$. We use the letter B here to avoid mixing it up with any of the velocities. We plug this in and solve for $\boldsymbol{B}$. The masses of the particles are only introduced in part four on the problem set, but they are m_1 and m_2.

$$\boldsymbol{p}_1^* + \boldsymbol{p}_2^* = \boldsymbol{0}$$

Write out the non-relativistic momentum.

$$\iff m_1 \boldsymbol{v}_1^* + m_2 \boldsymbol{v}_2^* = \boldsymbol{0}$$

Insert the transformation. $\boldsymbol{v}_2 = \boldsymbol{0}$ because the in the laboratory frame, the second particle is initially at rest.

$$\iff m_1[\boldsymbol{v}_1 - \boldsymbol{B}] + m_2[-\boldsymbol{B}] = \boldsymbol{0}$$
$$\iff [m_1 + m_2]\boldsymbol{B} = m_1 \boldsymbol{v}_1$$
$$\iff \boldsymbol{B} = \frac{m_1}{m_1 + m_2}\boldsymbol{v}_1$$

We will look at the edge cases here to make this more plausible. In the case $m_1 \gg m_2$, the center of mass frame will be the rest frame of the first particle. Therefore, the boost into the center of mass frame has to be almost the velocity of the particle itself. In case $m_1 \ll m_2$, the center of mass frame is pretty much identical to the lab frame. Therefore, the is no boost needed.

While we are at it, we will also derive the relation between $\boldsymbol{B}$ and $\boldsymbol{v}_1^*$ since we need that later on.

$$\boldsymbol{B} = \frac{m_1}{m_1 + m_2}\boldsymbol{v}_1$$

We boost into the moving system.

$$\iff \boldsymbol{B} = \frac{m_1}{m_1 + m_2}[\boldsymbol{B} + \boldsymbol{v}_1^*]$$
$$\iff \left[1 - \frac{m_1}{m_1 + m_2}\right]\boldsymbol{B} = \frac{m_1}{m_1 + m_2}\boldsymbol{v}_1^*$$
$$\iff \frac{m_2}{m_1 + m_2}\boldsymbol{B} = \frac{m_1}{m_1 + m_2}\boldsymbol{v}_1^*$$
$$\iff \boldsymbol{B} = \frac{m_1}{m_2}\boldsymbol{v}_1^*$$

This gives us the proportionality between $\boldsymbol{B}$ and $\boldsymbol{v}_1^*$.

1.2 Angular independence in center of mass frame

In the center of mass frame, there is zero total momentum. For two particles, like here, this means that they have opposing momentum. The energy is given by

$$E = \sum_i \frac{1}{2} m_i p_i^{*2}.$$

The total energy is conserved in an elastic collision,

$$\therefore \forall i, j : p_i^* = k_j^*.$$

That also means that there is no angular dependence, so there is no restriction in θ^*. This means that all of the four the momentum vectors lie on a circle centered at the origin.

1.3 Angular dependence in lab frame

We know that

$$\boldsymbol{p}_1 = \boldsymbol{k}_1 + \boldsymbol{k}_2$$

holds in the lab frame since the second particle was at rest initially. Then we can write down energy conservation:

$$m_1 p_1^2 = m_1 k_1^2 + m_2 k_2^2$$

Replace $\boldsymbol{k}_2$ with other knowns.

$$\iff m_1 p_1^2 = m_1 k_1^2 + m_2 [\boldsymbol{p}_1 - \boldsymbol{k}_1]^2$$

Expand.

$$\iff m_1 p_1^2 = m_1 k_1^2 + m_2 p_1^2 + m_2 k_1^2 - 2 m_2 \boldsymbol{p}_1 \boldsymbol{k}_1$$

Exchange on the sides.

$$\begin{aligned} \iff 2 m_2 \boldsymbol{p}_1 \boldsymbol{k}_1 &= m_1 k_1^2 + m_2 p_1^2 + m_2 k_1^2 - m_1 p_1^2 \\ \frac{\boldsymbol{p}_1 \boldsymbol{k}_1}{p_1 k_1} &= \frac{1}{2} \left[\frac{m_1}{m_2} \frac{k_1}{p_1} + \frac{p_1}{k_1} + \frac{k_1}{p_1} - \frac{m_1}{m_2} \frac{p_1}{k_1} \right] \\ &= \cos(\theta) \end{aligned}$$

So one can see that k_1 depends on θ. We could try to solve this equation for k_1, but that was not asked.

1.4 Equal mass

The energy has to transfer completely to particle 2. That also means that the momentum has to transfer completely. This gives us three equations,

$$\boldsymbol{p}_1 = \boldsymbol{k}_2, \quad \boldsymbol{p}_2 = \boldsymbol{k}_1 = \boldsymbol{0}, \quad m_1 p_1^2 + 0 = 0 + m_2 k_2^2.$$

From that, $m_1 = m_2$ follows pretty directly.

1.5 Relation of the angles

The angle in the lab frame is defined as

$$\cos(\theta) = \frac{\boldsymbol{p}_1 \boldsymbol{k}_1}{p_1 k_1}.$$

We can cancel m_1 twice and obtain the velocities.

$$= \frac{\boldsymbol{v}_1 \boldsymbol{w}_1}{v_1 w_1}$$

Those velocities can be boosted into the center of mass system.

$$= \frac{[\boldsymbol{v}_1^* + \boldsymbol{B}][\boldsymbol{w}_1^* + \boldsymbol{B}]}{|\boldsymbol{v}_1^* + \boldsymbol{B}||\boldsymbol{w}_1^* + \boldsymbol{B}|}$$

We expand.

$$= \frac{\boldsymbol{v}_1^* \boldsymbol{w}_1^* + \boldsymbol{v}_1^* \boldsymbol{B} + \boldsymbol{v}_1^* \boldsymbol{B} + B^2}{|\boldsymbol{v}_1^* + \boldsymbol{B}||\boldsymbol{w}_1^* + \boldsymbol{B}|}$$

The magnitude of the velocities of particle 1 is the same, since all momenta (particle 1, 2 and before/after) have the same magnitude. Divide by m_1 and obtain conserved velocities. The angle between $\boldsymbol{v}_1^*$ and $\boldsymbol{w}_1^*$ is θ^*. The boost $\boldsymbol{B}$ parallel to $\boldsymbol{v}_1^*$. That will let us express the scalar products in terms of that angle and the magnitudes of a single velocity.

$$= \frac{v_1^{*2} \cos(\theta^*) + v_1^* B + v_1^* B \cos(\theta^*) + B^2}{[v_1^* + B]\sqrt{v_1^{*2} + 2 v_1^* B \cos(\theta^*) + B^2}}$$

Now we use the proportionality between B and v_1^*, where we call the proportionality constant α.

$$= \frac{v_1^{*2} \cos(\theta^*) + \alpha v_1^{*2} + \alpha v_1^{*2} \cos(\theta^*) + \alpha^2 v_1^{*2}}{v_1^*[1+\alpha]\sqrt{v_1^{*2} + 2\alpha v_1^{*2} \cos(\theta^*) + \alpha^2 v_1^{*2}}}$$

We cancel off a v_1^{*2}.

$$= \frac{\cos(\theta^*) + \alpha + \alpha \cos(\theta^*) + \alpha^2}{[1+\alpha]\sqrt{1 + 2\alpha \cos(\theta^*) + \alpha^2}}$$

We factor the numerator.

$$= \frac{[1+\alpha][\cos(\theta^*) + \alpha]}{[1+\alpha]\sqrt{1 + 2\alpha \cos(\theta^*) + \alpha^2}}$$

Now we can cancel the first bracket.

$$= \frac{\alpha + \cos(\theta^*)}{\sqrt{1 + \alpha^2 + 2\alpha\cos(\theta^*)}}$$

Now since

$$\alpha = \frac{m_1}{m_2},$$

this works out by expanding the whole fraction with m_2.

$$= \frac{m_1 + m_2\cos(\theta^*)}{\sqrt{m_1^2 + m_2^2 + 2m_1m_2\cos(\theta^*)}}$$

In the limit $m_1 \ll m_2$, it simplifies to

$$\cos(\theta) = 1 \quad \Longleftarrow \quad \theta = 0.$$

In the limit $m_1 \gg m_2$, this simplifies to $\cos(\theta) = \cos(\theta^*)$ which is fulfilled given $\theta = \theta^*$.

When the masses are equal, we have

$$\begin{aligned}\cos(\theta) &= \frac{m_1 + m_1\cos(\theta^*)}{\sqrt{2m_1^2 + 2m_1^2\cos(\theta^*)}} \\ &= \frac{1 + \cos(\theta^*)}{\sqrt{2 + 2\cos(\theta^*)}} \\ &= \sqrt{\frac{1 + 1\cos(\theta^*)}{2}}\end{aligned}$$

Now we use the power reduction formula and get

$$= \cos\left(\frac{\theta^*}{2}\right).$$

This is consistent with

$$\theta = \frac{\theta^*}{2}.$$

1.6 Scattering cross section

We have a differential cross section given in the lab frame:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega^*} = |f(\theta^*, \phi^*)|^2.$$

From this, one can integrate over ϕ^* and remove that dependency to get the scattering cross section that depends on the angle θ^* only. This integral might be trivial if f does not depend on ϕ^*.

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\cos(\theta^*)} = \int_0^{2\pi} \mathrm{d}\phi^* \, |f(\theta^*, \phi^*)|^2$$

Assuming that this is a constant, we can use the chain rule:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\cos(\theta)} = \frac{\mathrm{d}\cos(\theta^*)}{\mathrm{d}\cos(\theta)} \frac{\mathrm{d}\sigma}{\mathrm{d}\cos(\theta^*)}$$

For this, equation (3) needs to be inverted and the derivative computed. We tried this, but the expression looked too strange to be correct, so we either made a mistake there, or this ansatz is incorrect.

2 Two-particle wave function

2.1 Normalization

We have to normalize the given function. We think that this N is meant to be the normalization, so we put this to the other side already.

$$\frac{1}{N^2} = \frac{1}{N^2} \int_{\mathbb{R}} \mathrm{d}x_1 \int_{\mathbb{R}} \mathrm{d}x_2 \, |\psi(x_1, x_2)|^2$$

So we put in the function.

$$= \int_{\mathbb{R}} \mathrm{d}x_1 \int_{\mathbb{R}} \mathrm{d}x_2 \, \exp\left(-2\frac{[x_1 - x_2]^2}{\sigma^2}\right) \exp\left(-2\frac{[x_1 + x_2]^2}{\Sigma^2}\right)$$

Now the best way is to do a coordinate transformation to parabolic coordinates (I think they are called that). This is just

$$\eta := x_1 - x_2, \quad \xi := x_1 + x_2.$$

The Jacobian of this transformation can be computed to be just 2. That transformation has the nice effect to decouple the two integrals.

$$= 2 \int_{\mathbb{R}} \mathrm{d}\xi \, \exp\left(-2\frac{\xi^2}{\sigma^2}\right) \int_{\mathbb{R}} \mathrm{d}\eta \, \exp\left(-2\frac{\eta^2}{\sigma^2}\right)$$

Those integrals can be solved using the solution formulas for the Gaussian integrals that we have missed somewhat in the last few homework problems. Having $a = 2/\sigma^2$, we get

$$= 2\sqrt{\frac{\pi}{2/\sigma^2}}\sqrt{\frac{\pi}{2/\Sigma^2}}$$

Then we just simplify and obtain

$$= \pi\sigma\Sigma.$$

2.2 Simple product

You can apply the binomial equation in the exponent and see that you will always have terms like $\exp(x_1 x_2)$ which cannot be factored into functions of one of the variables only.

2.3 Decomposition

Here we will calculate the Fourier transform of the given ψ. It then should fulfill the equation (4) from the problem set.

$$\tilde{\psi} = \mathscr{F}_{x_2}\mathscr{F}_{x_1}\psi(x_1, x_2)$$

We insert the integrals for the Fourier transform.

$$= \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{d}x_2 \int_{\mathbb{R}} \mathrm{d}x_2\, \psi(x_1, x_2) \exp(\mathrm{i}k_1 x_1) \exp(\mathrm{i}k_2 x_2)$$

We insert ψ.

$$= \frac{N}{2\pi}\int_{\mathbb{R}} \mathrm{d}x_2 \int_{\mathbb{R}} \mathrm{d}x_1\, \exp\left(-\frac{[x_1 - x_2]^2}{\sigma^2}\right) \exp\left(-\frac{[x_1 + x_2]^2}{\Sigma^2}\right) \exp(\mathrm{i}k_1 x_1) \exp(\mathrm{i}k_2 x_2)$$

The next thing to do is to convert this into a form where the beloved formulas can be used. First, we combine all the exponentials.

$$= \frac{N}{2\pi}\int_{\mathbb{R}} \mathrm{d}x_2 \int_{\mathbb{R}} \mathrm{d}x_1\, \exp\left(-\frac{[x_1 - x_2]^2}{\sigma^2} - \frac{[x_1 + x_2]^2}{\Sigma^2} + \mathrm{i}k_1 x_1 + \mathrm{i}k_2 x_2\right)$$

Then we expand the squares.

$$= \frac{N}{2\pi}\int_{\mathbb{R}} \mathrm{d}x_2 \int_{\mathbb{R}} \mathrm{d}x_1\, \exp\left(-\frac{x_1^2 - 2x_1 x_2 + x_2^2}{\sigma^2} - \frac{x_1^2 + 2x_1 x_2 + x_2^2}{\Sigma^2} + \mathrm{i}k_1 x_1 + \mathrm{i}k_2 x_2\right)$$

Then grouping by the powers of x_1 and x_2 is in order. We define

$$Z := \frac{1}{\sigma^2} + \frac{1}{\Sigma^2}, \quad Y := \frac{1}{\sigma^2} - \frac{1}{\Sigma^2}$$

and use that.

$$\begin{aligned}
\tilde{\psi} &= \frac{N}{2\pi} \int_{\mathbb{R}} \mathrm{d}x_2 \int_{\mathbb{R}} \mathrm{d}x_1 \, \exp\left(-Zx_1^2 + \left[\frac{2x_2}{\sigma^2} - \frac{2x_2}{\Sigma^2} + \mathrm{i}k_1\right] x_1 - Zx_2^2 + \mathrm{i}k_2 x_2\right) \\
&= \frac{N}{2\pi} \int_{\mathbb{R}} \mathrm{d}x_2 \int_{\mathbb{R}} \mathrm{d}x_1 \, \exp\left(-Zx_1^2 + [2Yx_2 + \mathrm{i}k_1] x_1 - Zx_2^2 + \mathrm{i}k_2 x_2\right)
\end{aligned}$$

The parts that do not depend on x_1 are moved in front of the integral for more clarity about the Gaussian integral.

$$= \frac{N}{2\pi} \int_{\mathbb{R}} \mathrm{d}x_2 \exp\left(-Zx_2^2 + \mathrm{i}k_2 x_2\right) \int_{\mathbb{R}} \mathrm{d}x_1 \, \exp\left(-Zx_1^2 + [2Yx_2 + \mathrm{i}k_1] x_1\right)$$

Using

$$a = Z, \quad b = 2Yx_2 + \mathrm{i}k_1$$

for the x_1 integral, we obtain using the Gaussian solution formula:

$$= \frac{N}{2\pi} \sqrt{\frac{\pi}{Z}} \int_{\mathbb{R}} \mathrm{d}x_2 \exp\left(-Zx_2^2 + \mathrm{i}k_2 x_2\right) \exp\left(\frac{1}{4Z} [2Yx_2 + \mathrm{i}k_1]^2\right)$$

Now all the x_1 are integrated over and gone. The same steps have to be done for the x_2 integral. So the exponential functions have to be combined and sorted by powers of x_2.

$$\begin{aligned}
&= \frac{N}{2\pi} \sqrt{\frac{\pi}{Z}} \int_{\mathbb{R}} \mathrm{d}x_2 \exp\left(-Zx_2^2 + \mathrm{i}k_2 x_2 + \frac{1}{4Z} [2Yx_2 + \mathrm{i}k_1]^2\right) \\
&= \frac{N}{2\pi} \sqrt{\frac{\pi}{Z}} \int_{\mathbb{R}} \mathrm{d}x_2 \exp\left(-Zx_2^2 + \mathrm{i}k_2 x_2 + \frac{1}{4Z} \left[4Y^2 x_2^2 + 4\mathrm{i}Yk_1 x_2 - k_1^2\right]\right) \\
&= \frac{N}{2\pi} \sqrt{\frac{\pi}{Z}} \int_{\mathbb{R}} \mathrm{d}x_2 \exp\left(-\left[Z + \frac{Y^2}{Z}\right] x_2^2 + \left[\mathrm{i}k_2 + \mathrm{i}\frac{Y}{Z}k_1\right] x_2 - \frac{k_1^2}{4Z}\right)
\end{aligned}$$

Everything that does not depend on x_2 is moved in front of the integral again.

$$= \frac{N}{2\pi} \sqrt{\frac{\pi}{Z}} \exp\left(-\frac{k_1^2}{4Z}\right) \int_{\mathbb{R}} \mathrm{d}x_2 \exp\left(-\left[Z + \frac{Y^2}{Z}\right] x_2^2 + \left[\mathrm{i}k_2 + \mathrm{i}\frac{Y}{Z}k_1\right] x_2\right)$$

Now we can apply the Gaussian integration again.

$$= \frac{N}{2\pi}\sqrt{\frac{\pi}{Z}}\sqrt{\frac{\pi}{Z+\frac{Y^2}{Z}}}\exp\left(-\frac{k_1^2}{4Z}\right)\exp\left(\frac{1}{4\left[Z+\frac{Y^2}{Z}\right]}\left[\mathrm{i}k_2+\mathrm{i}\frac{Y}{Z}k_1\right]^2\right)$$

We combine the square roots and the two exponentials.

$$= \frac{N}{2\sqrt{Z^2+Y^2}}\exp\left(-\frac{k_1^2}{4Z}+\frac{1}{4\left[Z+\frac{Y^2}{Z}\right]}\left[\mathrm{i}k_2+\mathrm{i}\frac{Y}{Z}k_1\right]^2\right)$$

We expand with Z^2 in the last fraction.

$$= \frac{N}{2\sqrt{Z^2+Y^2}}\exp\left(-\frac{1}{4Z}\left[k_1^2+\frac{[Yk_1+Zk_2]^2}{Z^2+Y^2}\right]\right)$$

We would expect this to be more symmetric in k_1 and k_2 since the original function was pretty symmetric in x_1 and x_2 as well.

Mathematica gives us

$$\tilde{\psi} = \frac{N\exp\left(\frac{1}{16}\left[2k_1k_2[\sigma^2-\Sigma^2]-k_1^2[\sigma^2+\Sigma^2]-k_2^2[\sigma^2+\Sigma^2]\right]\right)}{4\sqrt{\frac{1}{\sigma^2}+\frac{1}{\Sigma^2}}\sqrt{\frac{1}{\sigma^2+\Sigma^2}}},$$

which we can write more suggestive as

$$= \frac{N\exp\left(-\frac{1}{16}\left[[k_1-k_2]^2\sigma^2+[k_1+k_2]^2\Sigma^2\right]\right)}{4\sqrt{\frac{1}{\sigma^2}+\frac{1}{\Sigma^2}}\sqrt{\frac{1}{\sigma^2+\Sigma^2}}},$$

This is symmetric in k_1 and k_2, which we would expect since the original function was symmetric in its argument as well.

2.4 Use as boson or fermion function

In general, you just have to (anti)symmetrize the hell out the function and you get a completely (anti)symmetric function. In this particular case, the function already is symmetric in its coordinates. So it the antisymmetric part is zero. Therefore one can only use it as a boson function.

3 Totally symmetric N-particle state

The modulus squared of this thing can be written like this, where the dots mean every possible combination of all permutations with all permutations:

$$|S_+|i_1,i_2,\dots,i_N\rangle|^2 = \frac{1}{N}\left[\langle\dots|\dots\rangle+\langle\dots|\dots\rangle+\dots\right]$$

Assume first, that all i_j are different. Then there are $[N!]^2$ scalar products, and only $N!$ are nonzero, because the permutations are exactly the same for both terms. Therefore, the norm will be 1. Now assume that just one of the is has a multiplicity of n_i. Then terms that were orthogonal before, are now identical, giving a scalar product of 1. The number of possible permutations of this particular i is given by n_i. Therefore, the number of nonzero scalar products will go up by a factor n_i. This can be done for the other is as well, giving the normalization of

$$\prod_{i=0}^{N} n_i.$$