Disclaimer

This is a reviewed problem set for the module physics606.

This problem set was reviewed by a tutor. *This does not mean that it is a perfect solution. Neither I or the tutor imply that there are no further mistakes in this document.*

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physics606 – Advanced Quantum Theory

Problem Set 9

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problem number	achieved points	possible points
1	13	15
2	15	15
total	28	30

1 Expansion of a plane wave

1.1 Spherical Bessel functions

4/4

We start with the definition.

$$j_l(x) = -x^l \left[\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right]^l \frac{\sin(x)}{x}$$

Now we insert the series expansion of the sinc function.

$$= -x^l \left[\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right]^l \sum_{n=0}^{\infty} \frac{[-1]^n}{[2n+1]!} x^{2n}$$

Commuting terms.

$$= -\sum_{n=0}^{\infty} \frac{[-1]^n}{[2n+1]!} x^l \left[\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right]^l x^{2n}$$

The big square bracket will decrease the power of x^{2n} by $2l$ in total. Only terms with $n \geq l$ will contribute. We will omit higher terms. This leaves only one interesting term, we can drop the sum and set $n = l$.

$$= -\frac{[-1]^l}{[2l+1]!} x^l \left[\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right]^l x^{2l} + \mathcal{O}(x^{l+2})$$

The factors that we get by differentiating is every second of $2l, 2l-2, 2l-4, \ldots =: [2l]!!$.

✓

$$= -\frac{[-1]^l [2l]!!}{[2l+1]!} x^l + \mathcal{O}(x^{l+2})$$

Realizing that

$$[2l]!! = [2l][2l-2][2l-4]\ldots = 2l2[l-1]2[l-2]\ldots = 2^l l!$$

gives

✓

$$= [-1]^{l+1} \frac{2^l l!}{[2l+1]!} x^l + \mathcal{O}(x^{l+2})$$

That's because ~~ag~~ ... guess what? There is a mistake in the homework again :) defn. of $j_l(x) = (-x)^l \ldots$ not $-x^l$

This result differers from the result on the problem set by the l dependent sign.

1.2 Expression of x^l

3/3

We only look at the highest term in the Legendre polynomial:

$$P_l(x) = \frac{1}{2^l l!}\frac{\mathrm{d}^l}{\mathrm{d}x^l}\left[x^{2l} + \ldots\right] = \frac{1}{2^l l!}\frac{[2l]!}{l!} x^l + \ldots$$

The other terms contain lesser powers of x than x^l. Then this can be inverted to

✓

$$x^l = \frac{2^l [l!]^2}{[2l]!} P_l(x) + \ldots$$

Could be more explicit though :)

1.3 Use of orthogonality

2/2

We define $\xi := \cos(\theta)$ for this subsection. It is slightly confusing that k is used as the wave number and an index, we use m instead.

Start with expansion of plane wave.

$$\exp(\mathrm{i}kx\xi) = \sum_{l=0}^{\infty} A_l j_l(kx) P_l(\xi)$$

L^2 project this onto $P_m(\xi)$.

$$\int_{-1}^{1} \mathrm{d}\xi\, P_m(\xi) \exp(\mathrm{i}kx\xi) = \sum_{l=0}^{\infty} A_l j_l(kx) \int_{-1}^{1} \mathrm{d}\xi\, P_m(\xi) P_l(\xi)$$

Use orthogonality.

$$\int_{-1}^{1} \mathrm{d}\xi\, P_m(\xi) \exp(\mathrm{i}kx\xi) = \sum_{l=0}^{\infty} A_l j_l(kx) \frac{2}{2l+1} \delta_{lm}$$

Execute δ.

$$\int_{-1}^{1} \mathrm{d}\xi\, P_m(\xi) \exp(\mathrm{i}kx\xi) = A_m j_m(kx) \frac{2}{2k+1}$$

Move fraction to other side.

$$\frac{2m+1}{2} \int_{-1}^{1} \mathrm{d}\xi\, P_m(\xi) \exp(\mathrm{i}kx\xi) = A_m j_m(kx)$$

This is not the result on the problem set, since the integration measure $\mathrm{d}\cos(\theta) = \mathrm{d}\xi$ is missing there.

1.4 Extraction of A_l

5/6

We start with the expression we just derived.

$$A_l j_l(kx) = \frac{2l+1}{2} \int_{-1}^{1} \mathrm{d}\xi\, P_l(\xi) \exp(\mathrm{i}kx\xi)$$

We express j_l through its approximation around $kx = 0$ and express P_l in terms of x.

$$A_m [kx]^l \frac{2^l l!}{[2l+1]!} = \frac{2l+1}{2} \int_{-1}^{1} \mathrm{d}\xi\, \frac{[2l]!}{2^l [l!]^2} \xi^l \exp(\mathrm{i}kx\xi)$$

Next we insert the *definition* of the exponential.

$$A_m [kx]^l \frac{2^l l!}{[2l+1]!} = \frac{2l+1}{2} \int_{-1}^{1} \mathrm{d}\xi\, \frac{[2l]!}{2^l [l!]^2} \xi^l \left[1 + \mathrm{i}kx\xi + \mathcal{O}(\xi^2)\right]$$

We move the Landau $\mathcal{O}$ out of the integral.

$$A_m [kx]^l \frac{2^l l!}{[2l+1]!} = \frac{2l+1}{2} \int_{-1}^{1} \mathrm{d}\xi\, \frac{[2l]!}{2^l [l!]^2} \xi^l [1 + \mathrm{i}kx\xi] + \mathcal{O}(\xi^{l+2})$$

Move the factors to the right hand side.

$$A_m = \frac{[2l+1]!}{2^l l!} \frac{2l+1}{2} \frac{1}{[kx]^l} \frac{[2l]!}{2^l [l!]^2} \int_{-1}^{1} \mathrm{d}\xi \, \xi^l [1 + \mathrm{i}kx\xi] + \mathcal{O}(\xi^{l+2})$$

Simplify.

$$A_m = [2l+1]![2l+1] \frac{[2l]!}{2^{2l+1}[l!]^3} \frac{1}{[kx]^l} \int_{-1}^{1} \mathrm{d}\xi \, \xi^l [1 + \mathrm{i}kx\xi] + \mathcal{O}(\xi^{l+2})$$

$$A_m = [2l+1]^2 \frac{[[2l]!]^2}{2^{2l+1}[l!]^3} \frac{1}{[kx]^l} \int_{-1}^{1} \mathrm{d}\xi \, \xi^l [1 + \mathrm{i}kx\xi] + \mathcal{O}(\xi^{l+2})$$

One of the integrals will give zero, the other will contribute something that only depends on kx.

2 Scattering on a dipole

2.1 Scattering amplitude

4/4

Figure 1 shows the two scattering centers as well as the incoming and outgoing wave vectors. The Δs and $\Delta s'$ are the path differences for the two vectors. The incident waves are in phase and have the same wave vector $\boldsymbol{k}$. The incoming waves, represented by two wave vectors, scatter at the respective positions. At the point where they scatter, they have a phase shift due to the path difference. The path difference Δs is the projection of $\boldsymbol{d}$ onto the direction of $\boldsymbol{k}$, namely

$$\Delta s = \boldsymbol{d} \frac{\boldsymbol{k}}{k}.$$

The phase shift is this distance times the wave number, so the phase difference is given by the scalar product

$$\Delta \phi = \boldsymbol{dk}.$$

A similar thing happens for the scattered waves, they also have a path difference. In the case where $\boldsymbol{k}$ and $\boldsymbol{d}$, as well as $\boldsymbol{k}'$ and $\boldsymbol{d}$ point into the same direction (such that $\boldsymbol{kd} > 0 \wedge \boldsymbol{k}'\boldsymbol{d} > 0$), like in the picture, the path difference has a negative sign now. For the incoming wave, the right wave had an additional path, now it is the left wave with the additional path. We count this negatively then. The phase shift here is

$$\Delta \phi' = -\boldsymbol{dk}'.$$

The total phase shift therefore is

$$\boldsymbol{d}[\boldsymbol{k} - \boldsymbol{k}'] = \boldsymbol{dq}.$$

Equation (10) on the problem set gives the monopole scattering amplitude. It depends on the

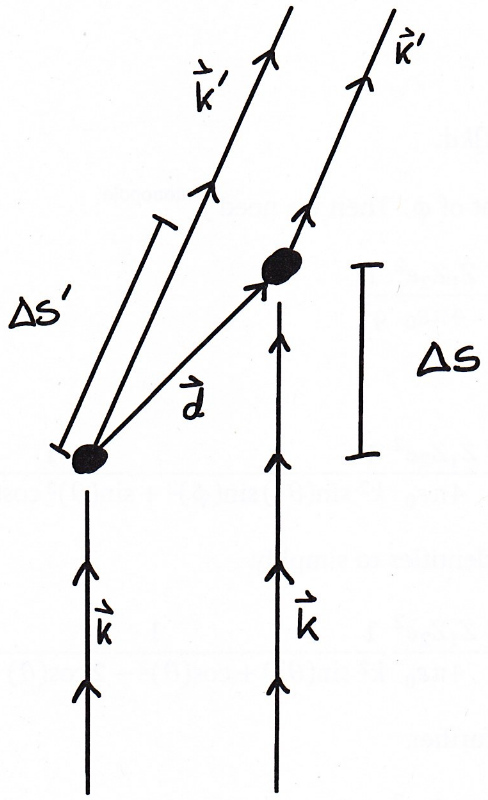

Figure 1: Path difference for the two scattered waves.

charge of the particles. Since the second particle has opposite charge, we will get the negative of the amplitude. Adding both amplitudes with the phase shift gives us the desired

$$f_{\boldsymbol{k}}^{\text{dipole}} = [1 - \exp(-i\boldsymbol{q}\boldsymbol{d})] f_{\boldsymbol{k}}^{\text{monopole}}.$$ neat

2.2 Case of parallel incident

3/3

We have

$$\boldsymbol{d} = \begin{pmatrix} 0 \\ 0 \\ d \end{pmatrix}.$$

Then

$$\boldsymbol{q}\boldsymbol{d} = [1 - \cos(\theta)]kd.$$

This part is independent of ϕ. Then we need $f_{\boldsymbol{k}}^{\text{monopole}}$:

$$f_{\boldsymbol{k}}^{\text{monopole}} = -\frac{2M}{\hbar}\frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0}\frac{1}{q^2}$$

We insert $\boldsymbol{q}$.

$$= -\frac{2M}{\hbar}\frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0}\frac{1}{k^2}\frac{1}{\sin(\theta^2)\sin(\phi)^2 + \sin(\theta)^2\cos(\phi)^2 + [1-\cos(\theta)]^2}$$

We use trigonometric identities to simplify.

$$= -\frac{2M}{\hbar}\frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0}\frac{1}{k^2}\frac{1}{\sin(\theta^2) + \cos(\theta)^2 - 2\cos(\theta) + 1}$$

This can be simplified further.

$$= -\frac{2M}{\hbar}\frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0}\frac{1}{k^2}\frac{1}{2 - 2\cos(\theta)}$$

For convenience, we introduce $\xi := 1 - \cos(\theta)$.

$$= -\frac{2M}{\hbar}\frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0}\frac{1}{k^2}\frac{1}{2\xi}$$

For even more convenience, we introduce A to carry all the constant factors.

$$= \frac{A}{k^2\xi}$$

The whole scattering amplitude then is

$$f_k^{\text{dipole}} = \frac{A}{k^2}\frac{1-\exp(-\mathrm{i}\xi kd)}{\xi}.$$

We write down the leading terms of the exponential function.

$$= \frac{A}{k^2}\frac{\mathrm{i}\xi kd + \mathcal{O}(\xi^2)}{\xi}$$

We cancel ξ.

$$= \frac{A}{k^2}\mathrm{i}kd + \mathcal{O}(\xi)$$

This is now well behaved in the limit $\theta \to 0$ implying $\xi \to 0$. It will be a finite nonzero value. The modulus squared of this has the same properties. Therefore the differential cross section is also zero and independent of ϕ. The integral over Ω does not change any of this. σ itself is always independent of any angles since those are integrated away.

2.3 Case of perpendicular incident

3/3

Now we have

$$\boldsymbol{d} = \begin{pmatrix} 0 \\ d \\ 0 \end{pmatrix}.$$

q^2 is left unchanged by this. The scalar product becomes

$$\boldsymbol{qd} = \sin(\theta)\cos(\phi)kd.$$

With that we get:

$$f_k^{\text{dipole}} = \frac{A}{2k^2}\frac{1-\exp(-\mathrm{i}\sin(\theta)\cos(\phi)kd)}{1-\cos(\theta)}$$

The differential cross section is given by its modulus squared:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left|f_k^{\text{dipole}}\right|^2.$$

We insert our previous result.

$$= \frac{A^2}{4k^4}\left|\frac{1-\exp(-\mathrm{i}\sin(\theta)\cos(\phi)kd)}{1-\cos(\theta)}\right|^2$$

The modulus squared is the same as the product with the complex conjugate.

$$= \frac{A^2}{4k^4} \frac{[1-\exp(-\mathrm{i}\sin(\theta)\cos(\phi)kd)][1-\exp(\mathrm{i}\sin(\theta)\cos(\phi)kd)]}{[1-\cos(\theta)]^2}$$

The two exponentials can be combined into a cosine.

$$= \frac{A^2}{4k^4} \frac{2-2\cos(\sin(\theta)\cos(\phi)kd)}{[1-\cos(\theta)]^2}$$

We cancel 4.

$$= \frac{A^2}{k^4} \frac{1-\cos(\sin(\theta)\cos(\phi)kd)}{[1-\cos(\theta)]^2}$$

This expression depends on ϕ, but diverges for $\theta \to 0$ since the numerator is ξ for small θ, but the denominator is ξ^2.

2.4 Large momentum transfer

5|5

We have $qd \gg 1$ here. The angle spanned between $\boldsymbol{q}$ and $\boldsymbol{d}$ shall be α which depends on θ in some way. Then the scattering amplitude is proportional to

$$1-\exp(-\mathrm{i}qd\cos(\alpha)).$$

We integrate the modulus squared of this over the interval $\theta \in [a-b, a+b]$ where a can be chosen in a way that α becomes θ. So we write the part of the cross section that we are interested in as

$$\tilde{\sigma} = \int_{a-b}^{a+b} \mathrm{d}\cos(\theta)\,[1-\exp(-\mathrm{i}qd\cos(\theta))][1-\exp(\mathrm{i}qd\cos(\theta))].$$

Now we out the integration and yield after a few steps of simplification:

$$= 4b + \frac{4}{qd}\exp(-\mathrm{i}qda)\sin(b).$$

For sufficiently large b, the first term will always be way greater than the second, since $qd \gg 1$ was given. Therefore, this will be $4b$.

When a single particle is looked at, the amplitude scaling factor is just 1. The integral over the same domain gives $2b$. For two independent particles, this is twice this, $4b$. Therefore, regardless of the choice of a, and with that independent of the orientation of $\boldsymbol{d}$ to $\boldsymbol{q}$, the cross section is like the incoherent sum.