Disclaimer

This is a problem set (as turned in) for the module physics606.

This problem set is not reviewed by a tutor. This is just what I have turned in.

All problem sets for this module can be found at
http://martin-ueding.de/de/university/msc_physics/physics606/.

[disclaimer]

physics606 – Advanced Quantum Theory

Problem Set 8

Martin Ueding
mu@martin-ueding.de

Lino Lemmer
l2@uni-bonn.de

2014-11-28
Group 2 – Dilege Gülmez

problem number	achieved points	possible points
1		14
2		15
total		29

1 Wave packet

We again write vectors in bold type, like $\boldsymbol{k}$, as opposed to scalars in regular italic font, like k. We also adopt the notation that $k = |\boldsymbol{k}|$.

1.1 Phase velocity

The phase of a single plane wave has to be constant, so

$$\boldsymbol{k}\boldsymbol{x} - \omega(k)t = c_1.$$

The total time derivative has to be zero, then. $\boldsymbol{k}$ is constant for a monochrome wave. This means

$$\begin{aligned} \boldsymbol{k}\dot{\boldsymbol{x}} &= \omega(k). \\ \iff k\frac{\boldsymbol{k}}{k}\dot{\boldsymbol{x}} &= \omega(k) \\ \iff \frac{\boldsymbol{k}}{k}\dot{\boldsymbol{x}} &= \frac{\omega(k)}{k} \end{aligned}$$

The part of the velocity that is in the direction of $\boldsymbol{k}$ is given by the right hand side. Since the propagation is taken to be this $\boldsymbol{k}$ direction, we can just write down the projection onto $\boldsymbol{k}$.

$$\iff \boldsymbol{v}_{\mathrm{ph}} = \frac{\boldsymbol{k}}{k}\frac{\omega(k)}{k}$$

1.2 Group velocity

We start by inserting $\boldsymbol{k} = \boldsymbol{k}_0 + \boldsymbol{\delta}$ into equation (1) from the problem set.

$$\psi(\boldsymbol{x}, t) = \int \frac{\mathrm{d}^3\delta}{[2\pi]^3} a(\boldsymbol{k}_0 + \boldsymbol{\delta}) \exp\left(\mathrm{i}\left[[\boldsymbol{k}_0 + \boldsymbol{\delta}]\boldsymbol{x} - \omega\left(|\boldsymbol{k}_0 + \boldsymbol{\delta}|\right) t\right]\right)$$

We expand the angular frequency around $\boldsymbol{\delta} = \boldsymbol{0}$:

$$\omega\left(|\boldsymbol{k}_0 + \boldsymbol{\delta}|\right) = \omega(k_0) + \frac{\mathrm{d}\omega}{\mathrm{d}k}\boldsymbol{\delta}\frac{\boldsymbol{k}_0}{k_0} + \mathcal{O}(\delta^2)$$

We insert this into the integral up to first order.

$$\approx \int \frac{\mathrm{d}^3\delta}{[2\pi]^3} a(\boldsymbol{k}_0 + \boldsymbol{\delta}) \exp\left(\mathrm{i}\left[[\boldsymbol{k}_0 + \boldsymbol{\delta}]\boldsymbol{x} - \left[\omega(k_0) + \frac{\mathrm{d}\omega}{\mathrm{d}k}\boldsymbol{\delta}\frac{\boldsymbol{k}_0}{k_0}\right] t\right]\right)$$

The parts that depend on $\boldsymbol{\delta}$ are taken out front.

$$= \exp\left(\mathrm{i}\left[\boldsymbol{k}_0\boldsymbol{x} - \omega(k_0)t\right]\right) \int \frac{\mathrm{d}^3\delta}{[2\pi]^3} a(\boldsymbol{k}_0 + \boldsymbol{\delta}) \exp\left(\mathrm{i}\left[\boldsymbol{\delta}\boldsymbol{x} - \frac{\mathrm{d}\omega}{\mathrm{d}k}\boldsymbol{\delta}\frac{\boldsymbol{k}_0}{k_0}t\right]\right)$$

Now we can do the same thing as in the first problem where we have factored out $\boldsymbol{k}$.

$$= \exp\left(\mathrm{i}\boldsymbol{k}_0\left[\boldsymbol{x} - v_{\mathrm{ph}}t\right]\right) \int \frac{\mathrm{d}^3\delta}{[2\pi]^3} a(\boldsymbol{k}_0 + \boldsymbol{\delta}) \exp\left(\mathrm{i}\left[\boldsymbol{\delta}\boldsymbol{x} - \frac{\mathrm{d}\omega}{\mathrm{d}k}\boldsymbol{\delta}\frac{\boldsymbol{k}_0}{k_0}t\right]\right)$$

The next thing is to factor out the $\boldsymbol{\delta}$.

$$= \exp\left(\mathrm{i}\boldsymbol{k}_0\left[\boldsymbol{x} - v_{\mathrm{ph}}t\right]\right) \int \frac{\mathrm{d}^3\delta}{[2\pi]^3} a(\boldsymbol{k}_0 + \boldsymbol{\delta}) \exp\left(\mathrm{i}\boldsymbol{\delta}\left[\boldsymbol{x} - \frac{\mathrm{d}\omega}{\mathrm{d}k}\frac{\boldsymbol{k}_0}{k_0}t\right]\right)$$

This is the form that we had to show for the given hint when equation (4) from the problem set it inserted.

$$= \exp\left(\mathrm{i}\boldsymbol{k}_0\left[\boldsymbol{x} - v_{\mathrm{ph}}t\right]\right) \int \frac{\mathrm{d}^3\delta}{[2\pi]^3} a(\boldsymbol{k}_0 + \boldsymbol{\delta}) \exp\left(\mathrm{i}\boldsymbol{\delta}\left[\boldsymbol{x} - v_{\mathrm{gr}}t\right]\right)$$

1.3 Time evolution

It is easiest to write this as ratio of the given ones. The only difference is the velocity in the first exponential function.

$$\frac{\psi(\boldsymbol{x},t)}{\psi(\boldsymbol{x}-\boldsymbol{v}_{\mathrm{gr}}t,0)} \approx \frac{\exp\left(\mathrm{i}\boldsymbol{k}_0\left[\boldsymbol{x}-v_{\mathrm{ph}}t\right]\right)\int\frac{\mathrm{d}^3\delta}{[2\pi]^3}a(\boldsymbol{k}_0+\boldsymbol{\delta})\exp\left(\mathrm{i}\boldsymbol{\delta}\left[\boldsymbol{x}-v_{\mathrm{gr}}t\right]\right)}{\exp\left(\mathrm{i}\boldsymbol{k}_0\left[\boldsymbol{x}-v_{\mathrm{gr}}t\right]\right)\int\frac{\mathrm{d}^3\delta}{[2\pi]^3}a(\boldsymbol{k}_0+\boldsymbol{\delta})\exp\left(\mathrm{i}\boldsymbol{\delta}\left[\boldsymbol{x}-v_{\mathrm{gr}}t\right]\right)}$$

This can be simplified since the integrals are just the exact same.

$$= \frac{\exp\left(\mathrm{i}\boldsymbol{k}_0\left[\boldsymbol{x}-v_{\mathrm{ph}}t\right]\right)}{\exp\left(\mathrm{i}\boldsymbol{k}_0\left[\boldsymbol{x}-v_{\mathrm{gr}}t\right]\right)}$$

The remainder is easy, just put the two exponentials together

$$= \exp\left(\mathrm{i}\boldsymbol{k}_0\left[\boldsymbol{v}_{\mathrm{gr}}-v_{\mathrm{ph}}\right]t\right)$$

This can then we rewritten in the form of equation (6) from the problem set.

1.4 Massless and massive particles

The relation between energy and angular frequency is given by

$$E(k) = \hbar\omega(k).$$

1.4.1 Massless particle

We have

$$\omega = ck$$

in this case. Therefore

$$v_{\mathrm{ph}} = v_{\mathrm{gr}} = c$$

which is not really surprising for a massless particle which has to propagate with the speed of light regardless of this energy.

1.4.2 Massive particle

Here

$$E = \frac{\hbar^2k^2}{2m}$$

such that

$$\omega = \frac{\hbar k^2}{2m}.$$

Then

$$v_{\text{ph}} = \frac{\hbar k}{2m}, \quad v_{\text{gr}} = \frac{\hbar k}{m}.$$

When this is inserted into the phase factor, one gets the desired result after a few simplifications which are hardly worth mentioning here.

1.5 Relativistic particle

The angular frequency is

$$\omega(k) = \frac{1}{\hbar}\sqrt{M^2c^4 + \hbar^2k^2c^2}.$$

From this, computing the velocities can be derived without new principles:

$$v_{\text{ph}} = \frac{1}{\hbar k}\sqrt{M^2c^4 + \hbar^2k^2c^2}, \quad v_{\text{gr}} = \frac{\hbar}{\sqrt{M^2c^4 + \hbar^2k^2c^2}}hc^2$$

Taking the limits is not really difficult since nothing diverges. We obtain

$$\lim_{M\to 0} v_{\text{ph}} = c, \quad \lim_{M\to 0} v_{\text{gr}} = c, \quad \lim_{k\to 0} v_{\text{gr}} = 0.$$

2 Scattering on a constant potential

2.1 Scattering amplitude

Given is

$$f_{\boldsymbol{k}}(\theta) = -\frac{2MV_0}{q\hbar^2}\int_0^\infty \mathrm{d}r'\, r\sin(qr')$$

and the following potential:

$$V(\boldsymbol{x}) = \begin{cases} V_0 & x < r_0 \\ 0 & \text{else} \end{cases}$$

In this part of the problem, we have to compute $f_{\boldsymbol{k}}$ explicitly.

Well, this does not make too much sense as the problem is posed. The integral does not converse, since the integrand oscillates badly, see figure 1. It is not *Mathematica* integrable either.

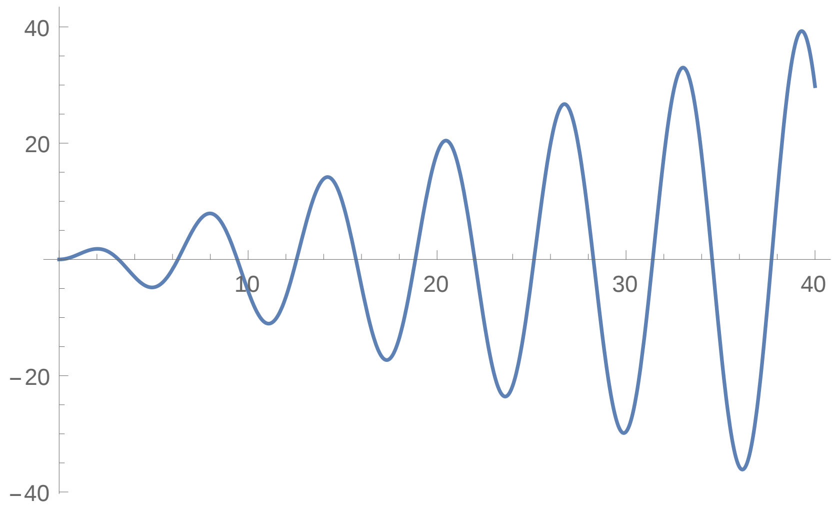

Figure 1: Plot of $r\sin(r)$.

We try to salvage this problem and assume that the V_0 should be a $V(x)$ within the integral. Then the only difference is that integration bounds are from 0 to r_0 instead of ∞, like so:

$$f_{\boldsymbol{k}}(\theta) = -\frac{2MV_0}{q\hbar^2} \int_0^{r_0} \mathrm{d}r'\, r \sin(qr').$$

Now this integral is actually solvable using partial integration. We obtain

$$f_{\boldsymbol{k}}(\theta) = \frac{2MV_0}{q\hbar^2} \left[\frac{r_0}{q} \cos(qr_0) + \frac{1}{q^2} \sin(qr_0) \right]$$

The limit $q \to 0$ can be done before and after the integration. Both give the same result:

$$\lim_{q\to 0} f_{\boldsymbol{k}}(\theta) = -\frac{2MV_0}{\hbar^2} \lim_{q\to 0} \frac{1}{q} \int_0^{r_0} \mathrm{d}r'\, r' \sin(qr')$$

We expand the sine.

$$= -\frac{2MV_0}{\hbar^2} \lim_{q\to 0} \frac{1}{q} \int_0^{r_0} \mathrm{d}r'\, r' \left[qr' + \frac{1}{6}[qr']^3 + \mathcal{O}(q^5) \right]$$

Cancel the q.

$$= -\frac{2MV_0}{\hbar^2} \lim_{q\to 0} \int_0^{r_0} \mathrm{d}r'\, r' \left[r' + \frac{1}{6} q^2 r'^3 + \mathcal{O}(q^4) \right]$$

Exchange limit and integration. This is allowed because the integrand is out of C^∞.

$$= -\frac{2MV_0}{\hbar^2} \int_0^{r_0} \mathrm{d}r'\, r' \lim_{q\to 0} \left[r' + \frac{1}{6} q^2 r'^3 + \mathcal{O}(q^4) \right]$$

The limit is without any problems.

$$= -\frac{2MV_0}{\hbar^2} \int_0^{r_0} \mathrm{d}r' \, r'^2$$

This integral is trivial.

$$= -\frac{2MV_0}{\hbar^2} \frac{1}{3} r_0^3$$
$$= -\frac{2MV_0}{3\hbar^2} r_0^3$$

Now this can be done the other way around as well. One just has to apply L'Hôspital's rule a couple times. We have tried this and got the same result.

Since the result is $\propto r_0^3$, our salvage of this is not too wrong.

2.2 Cross section

We computed the cross section by putting our result from the previous part of this problem into equation (7) from the problem set. Since the expression itself was not explicitly asked for, we do not show it here. Then we plotted this in a half logarithmic plot, see figure 2. There one can see the minima and maxima quite well.

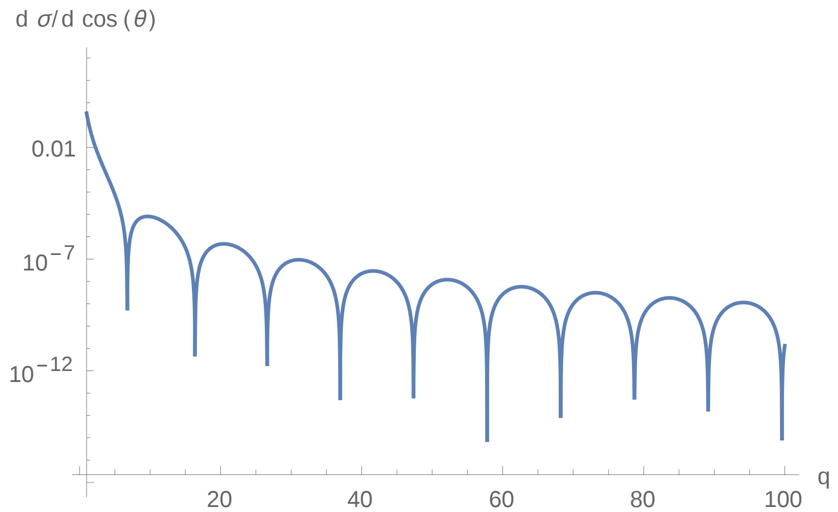

Figure 2: Half logarithmic plot of $\frac{\mathrm{d}\sigma}{\mathrm{d}\cos(\theta)}$ against q. All other constants are just set to unity for simplicity.

2.3 Born approximation

The wavefunctions are (Schwabl 2007, (18.9)):

$$\psi_{\boldsymbol{k}}(\boldsymbol{x}) = \mathrm{e}^{\mathrm{i}\boldsymbol{k}\boldsymbol{x}} + \frac{\mathrm{e}^{ikr}}{r} f_{\boldsymbol{k}}(\theta, \phi).$$

The Born approximation says that the scattered part only makes up a small part of the received wave. So we need

$$\begin{aligned} & \left| \frac{\mathrm{e}^{ikr}}{r_0} f_{\boldsymbol{k}}(\theta, \phi) \right| \ll 1 \\ \iff & \frac{|f_{\boldsymbol{k}}(\theta, \phi)|}{r_0} \ll 1 \\ \iff & |f_{\boldsymbol{k}}(\theta, \phi)| \ll r_0 \end{aligned}$$

For small q we already derived the limit. We just need to plug this in for f and get:

$$\left| \frac{2MV_0}{3\hbar^2} r_0^3 \right| \ll r_0 \iff \left| \frac{2MV_0}{3\hbar^2} r_0^2 \right| \ll 1.$$

References

Schwabl, Franz (2007). *Quantenmechanik*. 7. Berlin: Springer-Verlag. ISBN: 978-3-540-73674-5.