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[disclaimer]

physics606 – Advanced Quantum Theory

Problem Set 2

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problem number	achieved points	possible points
1		7
2		9
3		12
Total		28

1 Canonical Transformations and Classical Trajectories

1.1 Valid trajectories

Let $(q_i(t), p_i(t))$ be a solution to the equations of motion. Then for q_i we have

$$\dot{q}_i(t) = \left\{q_i(t), H\right\}.$$

We add the same to both sides which will not make any difference:

$$\iff \dot{q}_i(t) + \epsilon \left\{\frac{\partial g}{\partial p_i}, H\right\} = \left\{q_i(t), H\right\} + \epsilon \left\{\frac{\partial g}{\partial p_i}, H\right\}.$$

On the left hand side we expand the Poisson bracket into its definition.

$$\iff \dot{q}_i(t) + \epsilon \sum_j \left[\frac{\partial^2 g}{\partial q_j \partial p_i} \frac{\partial H}{\partial p_j} - \frac{\partial^2 g}{\partial p_j \partial p_i} \frac{\partial H}{\partial q_j}\right] = \left\{q_i(t) + \epsilon \frac{\partial g}{\partial p_i}, H\right\}.$$

The equations of motion allow us to eliminate H on the left hand side.

$$\iff \dot{q}_i(t) + \epsilon \sum_j \left[\frac{\partial^2 g}{\partial q_j \partial p_i} \dot{q}_j + \frac{\partial^2 g}{\partial p_j \partial p_i} \dot{p}_j \right] = \left\{ q_i(t) + \epsilon \frac{\partial g}{\partial p_i}, H \right\}.$$

The term in the square bracket is

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial}{\partial p_i} g\left(q_1(t), \dots, q_n(t), p_1(t), \dots, p_n(t)\right).$$

Therefore we can write this as

$$\iff \dot{q}_i(t) + \epsilon \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial g}{\partial p_i} = \left\{ q_i(t) + \epsilon \frac{\partial g}{\partial p_i}, H \right\}.$$

We factor out the time derivative and yield

$$\iff \frac{\mathrm{d}}{\mathrm{d}t} \left[q_i(t) + \epsilon \frac{\partial g}{\partial p_i} \right] = \left\{ q_i(t) + \epsilon \frac{\partial g}{\partial p_i}, H \right\}.$$

Inserting the definition of the transformation gives us

$$\iff \frac{\mathrm{d}}{\mathrm{d}t} \bar{q}_i(t) = \left\{ \bar{q}_i(t), H \right\}.$$

That is the equation of motion for $\bar{q}_i$. The same thing can be done with $\bar{p}_i$. Therefore, these infinitesimal transformations generate more valid trajectories.

1.2 Single particle

It is not clear to us what δ is (scalar constant, scalar function of something else, variational operator, ...). We assume that it is a simple scalar.

The generator is $g = \delta p_k$ since

$$\delta = \frac{\partial g}{\partial p_k}.$$

For this to be a canonical transformation, we have to have $\{g, H\} = 0$. That means

$$\{\delta p_k, H\} = \delta \dot{p}_k = -\frac{\partial H}{\partial q_k} = 0.$$

The hamiltonian must not depend on q_k for this to work. Also since $\dot{p}_k = 0$ this means that the momentum in the k-direction is preserved, as Noether's theorem also says.

2 Canonical Transformations in Quantum Mechanics

2.1 Transformation of the coefficients

Equation (3) from the problem set states:

$$|\psi\rangle = \sum_n c_n(t) |n\rangle ,$$

where we introduced $|n\rangle$ as the eigenstates to the operator $\hat{g}$. The eigenvalues of $\hat{g}$ are g_n, so we simply get:

$$|\tilde{\psi}\rangle = \sum_n c_n(t) \hat{U}_g(\xi) |n\rangle = \sum_n c_n(t) g_n |n\rangle .$$

Then $\tilde{c}_n(t) = c_n(t) g_n$ follows.

2.2 Rotations around z-axis

Let

$$g = \hat{L}_z = -\mathrm{i}\hbar \frac{\partial}{\partial \phi}$$

be the generator now. Then the unitary operator is

$$\hat{U}_{\hat{L}_z}(\xi) = \exp\left(-\xi \frac{\partial}{\partial \phi}\right).$$

We take an eigenfunction of $\hat{L}_z$ which is

$$\psi_m(\phi, t) = \exp(\mathrm{i}m\phi).$$

When we apply $\hat{U}_{\hat{L}_z}(\xi)$ to this eigenfunction, we get

$$\hat{U}_{\hat{L}_z}(\xi) \exp(\mathrm{i}m\phi) = \exp(-\xi \mathrm{i}\phi) \exp(\mathrm{i}m\phi).$$

That can be written as

$$\exp(-\xi \mathrm{i}\phi) \exp(\mathrm{i}m\phi) = \psi(\phi - \xi, t),$$

which is a rotation around the z-axis with the angle ξ. This indeed generates rotations.

2.3 Rotations of superpositions

2.3.1 Case fixed l and fixed m

Our state is

$$|\psi\rangle = |l, m\rangle$$

where $|l, m\rangle$ is an eigenstate of $\hat{L}^2$ and $\hat{L}_z$ which commutes with the former. The part that $\hat{L}_z$ acts upon is $\exp(\mathrm{i}m\phi)$, so the operator $\hat{U}_{\hat{L}_z}(\xi)$ will add a phase factor of $\exp(-\mathrm{i}m\xi)$ to the whole expression. This does *not* change the physical reality, since it is just a change of the *overall* phase of $|\psi\rangle$.

2.3.2 Case fixed l but different m

Now our state is

$$|\psi\rangle = \sum_m c_m(t) |l, m\rangle,$$

where the coefficients are for the different values of m. The new coefficients look like this:

$$\tilde{c}_m(t) = c_m(t) \exp(-\mathrm{i}m\phi).$$

The different summands of the superposition have a phase change which depends on m. So this is no overall phase change that could be factored out. This does change the physical reality.

2.3.3 Case different l and different m

This is similar to the above case. The states are now:

$$|\psi\rangle = \sum_{l,m} c_{lm}(t) |l, m\rangle,$$

The coefficients still transform in the same m dependence as before, altering more than the overall phase of $|\psi\rangle$. This also changes the physical reality.

3 Gauge Invariance in Classical Electrodynamics

3.1 Gauge invariance

We have, without explicit functional dependence:

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}, \quad \boldsymbol{E} = -\boldsymbol{\nabla} U = \dot{\boldsymbol{A}}$$

And the potentials transform like so:

$$\boldsymbol{A} \to \boldsymbol{A} + \boldsymbol{\nabla}\lambda, \quad U \to U + \dot{\lambda}$$

Now this is:

$$\boldsymbol{B} \to \boldsymbol{\nabla} \times [\boldsymbol{A} + \boldsymbol{\nabla}\lambda] = \boldsymbol{\nabla} \times \boldsymbol{A} + \boldsymbol{0} = \boldsymbol{B}$$

So $\boldsymbol{B}$ is unchanged. Now we transform the electric field,

$$\boldsymbol{E} \to -\boldsymbol{\nabla} U + \boldsymbol{\nabla}\dot{\lambda} - \dot{\boldsymbol{A}} - \boldsymbol{\nabla}\dot{\lambda} = \boldsymbol{E},$$

which is also unchanged.

All this can be written more elegantly using differential forms. First of all we combine U and $\boldsymbol{A}$ into a single four-vector $\boldsymbol{A}$. Using the flat isomorphism in our four dimensional Minkoswki manifold we introduce the 1-form $A := \boldsymbol{A}^\flat$. Then we have the field strength tensor, a 2-form, $F := \mathrm{d}A$. The gauge transformation is the addition of a gradient to the spacial components and a negative time derivative of the same entity to the temporal component. We can write this summand as the four-vector $S^i := \nabla^i \lambda$. It depends on the signature of the metric for the components S^μ to yield the right signs to be the summands for A^i and $U = A^0$. Now $S_i = \nabla_i \lambda$ is the component way of writing a 1-form as the gradient of a 0-form (a scalar, which is what λ is). We write this as $S = \mathrm{d}\lambda$. Now the gauge transformation is $A \to A + \mathrm{d}\lambda$. Since $F = \mathrm{d}A$ and $\mathrm{d}^2 = 0$, it can be seen elegantly that F is not changed in any way:

$$F = \mathrm{d}A \to \mathrm{d}A + \mathrm{d}^2\lambda = \mathrm{d}A = F.$$

3.2 Homogeneous Maxwell equations

The first equation is fulfilled since ϵ is antisymmetric whereas the $\boldsymbol{\nabla}$ commute with each other. Summation convention is implied.

$$\boldsymbol{\nabla}\boldsymbol{B} = \boldsymbol{\nabla}[\boldsymbol{\nabla} \times \boldsymbol{A}] = \epsilon_{ijk} \nabla_i \nabla_j A_k = 0.$$

The second equation is fulfilled since a gradient has no curl and the last two summands cancel each other.

$$\boldsymbol{\nabla} \times \boldsymbol{E} - \dot{\boldsymbol{B}} = -\boldsymbol{\nabla} \times \boldsymbol{\nabla} U + \boldsymbol{\nabla} \times \dot{\boldsymbol{A}} - \boldsymbol{\nabla} \times \dot{\boldsymbol{A}} = \boldsymbol{0} + \boldsymbol{0}.$$

Again, this can be written with differential forms, this being a simple $\mathrm{d}F = 0$. Since $F := \mathrm{d}A$ it follows that $\mathrm{d}^2 A = 0$. If you write this in components you get

$$\nabla_{[\alpha} F_{\beta\gamma]} = 0,$$

where we have use the antisymmetrization brackets in the indices. For all values for α, β, γ out of $\{0, 1, 2, 3\}$ you will get one of the four equations shown in the vectorial notation.

3.3 Lorentz gauge

This is again pretty laborious in the SI-system and with three-vectors. First of all, the given gauge condition is wrong. In the nice way of writing it, it is

$$\partial_\mu A^\mu = 0,$$

meaning that the four-divergence of $\boldsymbol{A}$ vanishes. The Coulomb gauge similarly is $\partial_i A^i = 0$, meaning that the three-divergence of $\boldsymbol{A}$ vanishes. With this notation, it is easy to see the analogy. However, this means that in the clumsy way this is

$$\nabla \boldsymbol{A} = -\mu_0 \varepsilon_0 \dot{U},$$

note the time derivative on the U!

We will start with the first inhomogeneous equation:

$$\begin{aligned} \nabla \boldsymbol{E} &= \frac{\varrho}{\varepsilon_0} \\ \iff \nabla \left[-\nabla U - \dot{\boldsymbol{A}} \right] &= \frac{\varrho}{\varepsilon_0} \\ \iff -\triangle U - \nabla \dot{\boldsymbol{A}} &= \frac{\varrho}{\varepsilon_0} \end{aligned}$$

Now using the gradient of the correct gauge condition gives us

$$\iff -\triangle U - \mu_0 \varepsilon_0 \dot{U} = \frac{\varrho}{\varepsilon_0}.$$

We define

$$\Box := \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \triangle$$

in the SI-system to write this easier:

$$\iff \Box U = \frac{\varrho}{\varepsilon_0}.$$

This equation only depends on $\varrho \propto J^0$ and $U \propto A^0$ now, it is decoupled from the spatial components.

Now we will tend to the second equation, the spatial part.

$$\begin{aligned} \nabla \times \boldsymbol{B} &= \mu_0 \boldsymbol{j} + \mu_0 \varepsilon_0 \dot{\boldsymbol{E}} \\ \iff \nabla \times \nabla \times \boldsymbol{A} &= \mu_0 \boldsymbol{j} + \mu_0 \varepsilon_0 \left[-\nabla \dot{U} - \dot{\boldsymbol{A}} \right] \\ \iff \nabla \times \nabla \times \boldsymbol{A} &= \mu_0 \boldsymbol{j} - \mu_0 \varepsilon_0 \nabla \dot{U} - \mu_0 \varepsilon_0 \dot{\boldsymbol{A}} \end{aligned}$$

The double curl can be expanded to $\nabla \nabla \boldsymbol{A} - \triangle \boldsymbol{A}$. This can be shown with the Levi-Civita symbol:

$$[\nabla \times \nabla \times \boldsymbol{A}]_i = \epsilon_{ijk} \partial_j \epsilon_{kmn} \partial_m A_n = [\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}] \partial_j \partial_m A_n = \partial_i \partial_j A_j - \partial_j \partial_j A_i = [\nabla \nabla \boldsymbol{A} - \triangle \boldsymbol{A}]_i.$$

We insert that into the equation now:

$$\iff \nabla \nabla \boldsymbol{A} - \triangle A = \mu_0 \boldsymbol{j} - \mu_0 \varepsilon_0 \nabla \dot{U} - \mu_0 \varepsilon_0 \dot{\boldsymbol{A}}$$

The correct gauge condition gives us $\mu_0\varepsilon_0\boldsymbol{\nabla}\dot{U} - \boldsymbol{\nabla}\boldsymbol{\nabla}A$ which we use to replace any temporal-like quantities from the equation.

$$\iff \boldsymbol{\nabla}\boldsymbol{\nabla}\boldsymbol{A} - \triangle A = \mu_0 \boldsymbol{j} + \boldsymbol{\nabla}\boldsymbol{\nabla}\boldsymbol{A} - \mu_0\varepsilon_0\dot{\boldsymbol{A}}$$
$$\iff \mu_0\varepsilon_0\dot{\boldsymbol{A}} - \triangle A = \mu_0 \boldsymbol{j}$$

Using the definition of $\Box$ again here, we obtain

$$\iff \Box \boldsymbol{A} = \mu_0 \boldsymbol{j}.$$

Setting $A^0 := cU$ and $J^0 = \varrho/c$, we can write this equation easier as

$$\Box \boldsymbol{A} = \mu_0 \boldsymbol{J}.$$

This could even be written with differential forms:

$$\mathrm{d}^\star F = 4\pi^\star J,$$

which might be the most elegant form.

3.4 Gauge invarance of the Lagrangian

The difference of L and its transformed one is

$$\Delta L = \int_{\mathbb{R}^3} \mathrm{d}^3x \, [\varrho\dot{\lambda} + \boldsymbol{j}\boldsymbol{\nabla}\lambda].$$

Please note the difference between the Laplacian $\triangle$ and the “difference Delta” Δ. As it currently stands, that is not the total time derivative of another function. Therefore, this is not necessarily gauge invariant.

The action is the time integral of the Lagrange function, so we get:

$$\Delta S = \int_{\mathbb{R}} \mathrm{d}t \, \Delta L$$
$$= \int_{\mathbb{R}^4} \mathrm{d}t \; \mathrm{d}^3x \, [\varrho\dot{\lambda} + \boldsymbol{j}\boldsymbol{\nabla}\lambda]$$

This will be much clearer in four dimensional notation.

$$= \int_{\mathbb{R}^4} \mathrm{d}^4x \, [\varrho\dot{\lambda} + \boldsymbol{j}\boldsymbol{\nabla}] = \int_{\mathbb{R}^4} \mathrm{d}^4x \, J^\mu \partial_\mu \lambda$$

The continuity equation states that $\partial_\mu J^\mu = 0$, so we can write

$$\partial_\mu J^\mu \lambda = [\partial_\mu J^\mu]\lambda + J^\mu \partial_\mu \lambda = J^\mu \partial_\mu \lambda.$$

We can use this to write the integral as the volume integral over a divergence:

$$= \int_{\mathbb{R}^4} \mathrm{d}^4 x \, \partial_\mu J^\mu \lambda.$$

That integral can be transformed into a surface integral over $\boldsymbol{J}\lambda$. Using the assumption that the current $\boldsymbol{J}$ is bounded ($\exists R \in \mathbb{R}^+ \colon \forall r \geq R \colon |\boldsymbol{J}(r)| = 0$.) the whole term vanishes.

$$= 0$$

Therefore, the action is actually the same. Therefore, the physics do not change.