Disclaimer

This is a problem set (as turned in) for the module physics606.

This problem set is not reviewed by a tutor. This is just what I have turned in.

All problem sets for this module can be found at
http://martin-ueding.de/de/university/msc_physics/physics606/.

[disclaimer]

physics606 – Advanced Quantum Theory

Problem Set 1

Martin Ueding
mu@martin-ueding.de

Lino Lemmer
l2@uni-bonn.de

2014-10-14
Group 2 – Dilege Gülmez

problem number	achieved points	possible points
1		8
2		8
3		5
4		10
Total		31

I, Martin Ueding, would like to scan and upload the problem sets with your corrections to my website martin-ueding.de. There, the original problem set as well as the reviewed one will be licensed under the "Creative Commons Attribution-ShareAlike 4.0 International License". Is that okay with you?

Yes ☐ No ☐

1 Hermitean Operators

1.1 Real Eigenvalues

Let the eigenvalue of the operator Q on ψ_2 be q_2. For now, we assume $q_2 \in \mathbb{C}$. We start with the equation (1) from the problem set:

$$\int \mathrm{d}x\, \psi_1^*(x) Q \psi_2^*(x) = \int \mathrm{d}x\, \left[Q\psi_1(x)\right]^* \psi_2^*(x).$$

We expand the complex conjugate in the square bracket:

$$\int \mathrm{d}x\, \psi_1^*(x) Q \psi_2^*(x) = \int \mathrm{d}x\, \psi_1^*(x) Q^\dagger \psi_2^*(x).$$

The eigenvalue of $Q^\dagger$ is the complex conjugate of the eigenvalue of Q:

$$q_2 \int \mathrm{d}x\, \psi_1^*(x)\psi_2^*(x) = q_2^* \int \mathrm{d}x\, \psi_1^*(x)\psi_2^*(x)$$
$$\iff q_2 = q_2^*.$$

This restricts the eigenvalues to $\mathrm{Im}(q) = 0$ which means that $q \in \mathbb{R}$. The eigenvalues are all real.

1.2 Orthogonal Eigenfunctions

We will use the bra-ket notation here. Let $|n\rangle$ for $n \in \mathbb{N}$ (perhaps bounded to some N) be the eigenstates for Q with eigenvalues q_n. All the q_n are assume to be pairwise different.

First we have for $m, n \in \mathbb{N}$ which are assumed to be different:

$$\langle n|Q|m\rangle = q_m \langle n|m\rangle,$$
$$\langle m|Q|n\rangle = q_n \langle m|n\rangle.$$

Since

$$\langle n|Q|m\rangle = \langle m|Q|n\rangle^*,$$

the equations above are the complex conjugate of each other. This statement will be proven in the next part of this problem. Including the previously proven fact that the eigenvalues are real, we can derive the following equation:

$$q_m \langle n|m\rangle = q_n \langle n|m\rangle.$$

This leaves us with:

$$[q_m - q_n]\langle n|m\rangle = 0.$$

We assume non-degenerate eigenvalues, which means that $\langle n|m\rangle = 0$. The definition of orthogonality is a vanishing scalar product, which is the case here. Therefore $|n\rangle$ and $|m\rangle$ are orthogonal.

1.3 Matrix Representation

Basically we have to show that $Q_{ij} = Q^*_{ji}$ actually holds here. The definition of the matrix element is:

$$Q_{ij} = \int \mathrm{d}x \, \psi_i^*(x) Q \psi_j(x).$$

From equation (1) from the problem set we know that we can write this like this as well:

$$= \int \mathrm{d}x \, \left[Q^\dagger \psi_i(x)\right]^* \psi_j^*(x).$$

We include the last factor into the complex conjugation:

$$= \int \mathrm{d}x \, \left[\psi_j^*(x) Q^\dagger \psi_i(x)\right]^*.$$

Integration and complex conjugation commutes, leaving

$$= \left[\int \mathrm{d}x \, \psi_j^*(x) Q^\dagger \psi_i(x)\right]^*.$$

This, however, is just the definition of

$$= Q^*_{ji}.$$

2 Decomposition of a Wave Function

2.1 Coefficients

We have

$$\psi(x,t) = \sum_n u_n(t) \psi_n(x).$$

There ψ are different, one has an index, the other does not. We will use this notation although it is a little overloaded. From there, we project the ψ onto a ψ_m:

$$\begin{aligned} \int \mathrm{d}x \, \psi_n^*(x) \psi(x,t) &= \sum_m u_m(t) \int \mathrm{d}x \, \psi_n^*(x) \psi_m(x) \\ &= u_n(t). \end{aligned}$$

Therefore, the equation (4) from the problem set holds.

2.2 Normalization

For this problem, we just insert the definition of ψ:

$$\begin{aligned}\int \mathrm{d}x\, |\psi(x,t)|^2 &= \int \mathrm{d}x \left[\sum_n u_n^*(t)\psi_n^*(x)\right]\left[\sum_m u_m(t)\psi_m(x)\right] \\ &= \sum_{n,m} u_n^*(t)u_m(t)\int \mathrm{d}x\, \psi_n^*(x)\psi_m(x)\end{aligned}$$

Eigenfunctions are orthogonal, so this reduces to the $m = n$ terms:

$$\begin{aligned}&= \sum_{n,m} u_n^*(t)u_m(t)\,\langle n|m\rangle \\ &= \sum_n |u_n(t)|^2\end{aligned}$$

Since ψ had to be normalized, the condition

$$\sum_n |u_n(t)|^2 = 1$$

follows.

2.3 Expectation value

Just inserting again:

$$\begin{aligned}\langle Q\rangle &= \int \mathrm{d}x\, \psi^*(x)Q\psi(x) \\ &= \sum_{n,m} u_m^*(t)u_n(t)\int \mathrm{d}x\, \psi_m^*(x)Q\psi_n(x) \\ &= \sum_{n,m} u_m^*(t)u_n(t)q_n\delta_{nm} \\ &= \sum_n |u_n(t)|^2 q_n.\end{aligned}$$

3 Angular Momentum Operator

3.1 Eigenvalue

We have

$$L_z = -\mathrm{i}\hbar \frac{\partial}{\partial \phi}$$

and

$$\psi_m(\phi) = \frac{1}{\sqrt{2\pi}} \exp\left(\mathrm{i}m\phi\right).$$

First the eigenvalues:

$$\begin{aligned} L_z \psi_m(\phi) &= -\mathrm{i}\hbar \frac{\partial}{\partial \phi} \left[\frac{1}{\sqrt{2\pi}} \exp\left(\mathrm{i}m\phi\right) \right] \\ &= -\mathrm{i}\hbar \frac{\mathrm{i}m}{\sqrt{2\pi}} \exp\left(\mathrm{i}m\phi\right) \\ &= \hbar m \psi_m(\phi). \end{aligned}$$

Now we show, that the eigenfunctions are normalized:

$$\begin{aligned} \left|\psi_m\right|^2 &= \int_0^{2\pi} \mathrm{d}\phi \; \psi_m^* \psi_m \\ &= \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\phi \; \exp\left(-\mathrm{i}m\phi\right) \exp\left(\mathrm{i}m\phi\right) \\ &= \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\phi \\ &= 1. \end{aligned}$$

3.2 Integer m

We want

$$\psi_m(\phi) = \psi_m(\phi + 2\pi).$$

This implies

$$\begin{aligned} \exp(\mathrm{i}m\phi) &= \exp\left(\mathrm{i}m(\phi + 2\pi)\right) \\ &= \exp(\mathrm{i}m\phi) \underbrace{\exp(\mathrm{i}m2\pi)}_{=1, \text{ for } m\in\mathbb{N}}. \end{aligned}$$

Therefor m has to be integer.

3.3 Orthonormal eigenfunctions

$$\int_0^{2\pi} \mathrm{d}\phi\ \psi_l^* \psi_m = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\phi\ \exp(-\mathrm{i}l\phi)\exp(\mathrm{i}m\phi)$$
$$= \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\phi\ \exp\big(\mathrm{i}\phi(m-l)\big).$$

We look at $m \neq l$:

$$= \frac{1}{2\pi} \frac{1}{\mathrm{i}(m-l)} \Big[\exp\big(\mathrm{i}\phi(m-l)\big)\Big]_0^{2\pi},$$

from $m - l \in \mathbb{N}$ we get

$$= 0.$$

The other case $l = m$ gives because of the normalization

$$\int_0^{2\pi} \mathrm{d}\phi\ \psi_m^* \psi_m = 1.$$

These two cases show us, that the eigenfunctions are orthonormal:

$$\int_0^{2\pi} \mathrm{d}\phi\ \psi_l^* \psi_m = \delta_{lm}.$$

4 Canonical Transformations

4.1 Form-Invariance of Equations of Motion

See section 4.3 for the proof that Poisson brackets of arbitrary functions that are defined on the configuration manifold are invariant under canonical transformations. We will now use this to show the invariance of the equations of motions.

The time evolution can be expressed with the Poisson brackets:

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad \wedge \quad \dot{p}_i = \frac{\partial H}{\partial q_i} \qquad \Longleftrightarrow \qquad \dot{q}_i = \{q_i, H\} \quad \wedge \quad \dot{p}_i = \{p_i, H\}$$

To use the proof from section 4.3, we need to have two arbitrary functions. We chose them to be (q_i, H) for the first equation of motion and (p_i, H) for the second. Since both cases work similarly, we only show the first one here.

$$\dot{q}_i = \{q_i, H\}_{q,p}$$

Using the theorem:

$$= \left[\bar{q}_i, \bar{H}\right]_{\bar{q},\bar{p}}$$

From this, we can obtain

$$\dot{\bar{q}}_i = \frac{\partial \bar{H}}{\partial \bar{p}_i}.$$

If we adopt the sloppy notation from the exercise sheet, we get the same H:

$$\dot{\bar{q}}_i = \frac{\partial H}{\partial \bar{p}_i}.$$

4.2 Check for canonicality

To verify that it is a canonical transformation we have to calculate the Poisson brackets. This is a mechanical task, so we did this with *Mathematica*:

```
In[1]:= qbar[q_, p_] := Log[Sin[p]/q]

In[2]:= pbar[q_, p_] := q Cot[p]

In[3]:= poisson[a_, b_] := D[a[q, p], q] D[b[q, p], p] - D[a[q, p], p] D[b[q, p], q]

In[6]:= poisson[qbar, qbar] // Simplify
Out[6]= 0

In[7]:= poisson[pbar, pbar] // Simplify
Out[7]= 0

In[8]:= poisson[qbar, pbar] // Simplify
Out[8]= 1
```

The three relations hold, therefore the transformation is canonical.

4.3 Invariance of Poisson brackets

As the problem is stated, we believe that it actually is trivial. It says that one should proof that

$$\left\{A(q,p), B(q,p)\right\}_{q,p} = \left\{A(\bar{q},\bar{p}), B(\bar{q},\bar{p})\right\}_{\bar{q},\bar{p}}$$

holds. We mean this in the sense that both sides contain the functions A and B which take two arguments. Both sides only depend on dummy variables, the q and p with and without a bar. Since

$f(x) = f(y)$ if $x = y$ (as is the definition of a function), there cannot be any difference whether the Poisson brackets are spelled out with q or $\bar{q}$. That said, we think that the problem actually implies that the left and the right A are two different functions. So we will call them A and $\bar{A}$ to make this a non-trivial problem.

We now assume that the functions A and $\bar{A}$ relate in the following way:

$$\bar{A}\big(\bar{q}(q,p), \bar{p}(q,p)\big) = A(q,p).$$

So for a given q and p both functions take the same *value*, although the arguments for $\bar{A}$ and A differ. If this is to be the case, the total differential with respect to q (and p) has to be the same. This brings us to

$$\frac{\partial \bar{A}}{\partial \bar{q}} \frac{\partial \bar{q}}{\partial q} + \frac{\partial \bar{A}}{\partial \bar{p}} \frac{\partial \bar{p}}{\partial q} = \frac{\partial A}{\partial q}.$$

We start off with the Poisson bracket of A and B. Since there might be more than one coordinate, we start indexing them again.

$$\{A, B\}_{q,p} = \sum_k \left[\frac{\partial A}{\partial q_k} \frac{\partial B}{\partial p_k} - \frac{\partial A}{\partial p_k} \frac{\partial B}{\partial q_k} \right]$$

We use the relation that we derived using the total differential to expand the partial differentials. This is going to be messy.

$$\begin{aligned} &= \sum_{j,k,l} \left[\frac{\partial \bar{A}}{\partial \bar{q}_j} \frac{\partial \bar{q}_j}{\partial q_k} + \frac{\partial \bar{A}}{\partial \bar{p}_j} \frac{\partial \bar{p}_j}{\partial q_k} \right] \left[\frac{\partial \bar{B}}{\partial \bar{q}_l} \frac{\partial \bar{q}_l}{\partial p_k} + \frac{\partial \bar{B}}{\partial \bar{p}_l} \frac{\partial \bar{p}_l}{\partial p_k} \right] \\ &\quad - \sum_{j,k,l} \left[\frac{\partial \bar{A}}{\partial \bar{q}_j} \frac{\partial \bar{q}_j}{\partial p_k} + \frac{\partial \bar{A}}{\partial \bar{p}_j} \frac{\partial \bar{p}_j}{\partial p_k} \right] \left[\frac{\partial \bar{B}}{\partial \bar{q}_l} \frac{\partial \bar{q}_l}{\partial q_k} + \frac{\partial \bar{B}}{\partial \bar{p}_l} \frac{\partial \bar{p}_l}{\partial q_k} \right] \end{aligned}$$

We now expand this whole thing, such that it becomes even messier.

$$\begin{aligned} = \sum_{j,k,l} \Bigg[& \frac{\partial \bar{A}}{\partial \bar{q}_j} \frac{\partial \bar{q}_j}{\partial q_k} \frac{\partial \bar{B}}{\partial \bar{q}_l} \frac{\partial \bar{q}_l}{\partial p_k} + \frac{\partial \bar{A}}{\partial \bar{q}_j} \frac{\partial \bar{q}_j}{\partial q_k} \frac{\partial \bar{B}}{\partial \bar{p}_l} \frac{\partial \bar{p}_l}{\partial p_k} + \frac{\partial \bar{A}}{\partial \bar{p}_j} \frac{\partial \bar{p}_j}{\partial q_k} \frac{\partial \bar{B}}{\partial \bar{q}_l} \frac{\partial \bar{q}_l}{\partial p_k} \\ & + \frac{\partial \bar{A}}{\partial \bar{p}_j} \frac{\partial \bar{p}_j}{\partial q_k} \frac{\partial \bar{B}}{\partial \bar{p}_l} \frac{\partial \bar{p}_l}{\partial p_k} - \frac{\partial \bar{A}}{\partial \bar{q}_j} \frac{\partial \bar{q}_j}{\partial p_k} \frac{\partial \bar{B}}{\partial \bar{q}_l} \frac{\partial \bar{q}_l}{\partial q_k} - \frac{\partial \bar{A}}{\partial \bar{q}_j} \frac{\partial \bar{q}_j}{\partial p_k} \frac{\partial \bar{B}}{\partial \bar{p}_l} \frac{\partial \bar{p}_l}{\partial q_k} \\ & - \frac{\partial \bar{A}}{\partial \bar{p}_j} \frac{\partial \bar{p}_j}{\partial p_k} \frac{\partial \bar{B}}{\partial \bar{q}_l} \frac{\partial \bar{q}_l}{\partial q_k} - \frac{\partial \bar{A}}{\partial \bar{p}_j} \frac{\partial \bar{p}_j}{\partial p_k} \frac{\partial \bar{B}}{\partial \bar{p}_l} \frac{\partial \bar{p}_l}{\partial q_k} \Bigg] \end{aligned}$$

We will now group them again by the derivatives of $\bar{A}$ and $\bar{B}$ into four groups. The brackets that form from two of the above terms are a Poisson bracket of the transformed coordinates. To make it obvious which bracket there is, we added them to the left side.

$$= \sum_{j,k,l} \frac{\partial \bar{A}}{\partial \bar{q}_j} \frac{\partial \bar{B}}{\partial \bar{q}_l} \left[\frac{\partial \bar{q}_j}{\partial q_k} \frac{\partial \bar{q}_l}{\partial p_k} - \frac{\partial \bar{q}_j}{\partial p_k} \frac{\partial \bar{q}_l}{\partial q_k} \right] \qquad [\ldots] = \left\{ \bar{q}_j, \bar{q}_k \right\}_{q,p} = 0$$

$$+ \sum_{j,k,l} \frac{\partial \bar{A}}{\partial \bar{p}_j} \frac{\partial \bar{B}}{\partial \bar{p}_l} \left[\frac{\partial \bar{p}_j}{\partial q_k} \frac{\partial \bar{p}_l}{\partial p_k} - \frac{\partial \bar{p}_j}{\partial p_k} \frac{\partial \bar{p}_l}{\partial q_k} \right] \qquad [\ldots] = \left\{ \bar{p}_j, \bar{p}_k \right\}_{q,p} = 0$$

$$+ \sum_{j,k,l} \frac{\partial \bar{A}}{\partial \bar{q}_j} \frac{\partial \bar{B}}{\partial \bar{p}_l} \left[\frac{\partial \bar{q}_j}{\partial q_k} \frac{\partial \bar{p}_l}{\partial p_k} - \frac{\partial \bar{q}_j}{\partial p_k} \frac{\partial \bar{p}_l}{\partial q_k} \right] \qquad [\ldots] = \left\{ \bar{q}_j, \bar{p}_k \right\}_{q,p} = \delta_{jl}$$

$$+ \sum_{j,k,l} \frac{\partial \bar{A}}{\partial \bar{p}_j} \frac{\partial \bar{B}}{\partial \bar{q}_l} \left[\frac{\partial \bar{p}_j}{\partial q_k} \frac{\partial \bar{q}_l}{\partial p_k} - \frac{\partial \bar{p}_j}{\partial p_k} \frac{\partial \bar{q}_l}{\partial q_k} \right] \qquad [\ldots] = \left\{ \bar{p}_j, \bar{q}_k \right\}_{q,p} = -\delta_{jl}$$

The brackets reduce the whole expression considerably, we yield

$$= \sum_j \left[\frac{\partial \bar{A}}{\partial \bar{q}_j} \frac{\partial \bar{B}}{\partial \bar{p}_j} - \frac{\partial \bar{A}}{\partial \bar{p}_j} \frac{\partial \bar{B}}{\partial \bar{q}_j} \right]$$

which is just the definition of the following Poisson bracket:

$$= \left\{ \bar{A}, \bar{B} \right\}_{\bar{q},\bar{p}} .$$

This proves that the Poisson bracket of any function is invariant under canonical transformations. The function has to be transformed as well, though, unlike the notation on the problem set suggests.