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Geometry in Physics

Problem Set 11

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Group 1 – Jens Boos

problem	achieved points	possible points
Lie algebra of matrix groups	29	30
Field matter coupling in quantum mechanics	17	20
Total	46	50

1. Lie algebra of matrix groups

Are we talking about the abstract algebra of the groups or rather a matrix representation in a given dimension? The groups we look at are matrix groups of a given dimension. They have a "natural" matrix representation, which is the one used before generalizing those matrices to a group. The algebra can be written down in terms of those matrices as well. We will start with the matrix representation in n dimensions of the groups as well.

The basis of the Lie algebra are the generators, and we will take the physicist's convention of hermitian generators for compact groups here. We assume that the elements of the group are always exponentiated from some algebra element, i.e. that they are connected to the identity element e. ✓ Then we can just take the first order derivative at the identity, which means that we look at $T_e G$ which will give us $\mathfrak{g}$.

1.a. General linear group 4/5

→ no: only invertible n×n matrices!

The general linear group contains literally all matrices with n dimensions over the given field, which is $\mathbf{R}$ here. We can parameterize any matrix by its components. The matrix is given by:

$$A\left(\{\alpha^i{}_j\}_{ij}\right) = \{\alpha^i{}_j\}_{ij}.$$

The matrix is just given by the matrix of the parameters. ✓

We can obtain the generators:

$$T_i{}^j = -\mathrm{i}\frac{\partial A\left(\{\alpha^i{}_j\}_{ij}\right)}{\partial \alpha^i{}_j}(0) = -\mathrm{i}e_i \otimes e^j,$$

where we have used the explicit form of the matrices in the last step, of course. Those generators are not hermitian. The group in question is not compact, so this is normal.

The matrices only depend linearly on the parameters, so the generators already form the basis for the whole group as well. The algebra can then be written as

$$\mathfrak{gl}(n, \mathbf{R}) = \left\{-\mathrm{i}\alpha^i{}_j e_i \otimes e^j : \alpha^i{}_j \in \mathbf{R}\right\}.$$

From here, we can reach all the elements of the group with the exponential map:

$$g(\alpha) = \exp\left(\mathrm{i}\alpha^i{}_j T_i{}^j\right).$$ ✓

1.b. Special linear group 5/5

The special linear group has the constraint that the determinant has to be one. On the previous exercise sheet we derived the useful relation $\det \exp(A) = \exp(\operatorname{tr} A)$. In our case we can take $A \in \mathfrak{sl}$ and see that $\exp(A) \in \mathrm{GL}$, then. A unit determinant then corresponds to generators which do not have a trace. ✓

We remove the trace from the generators that we had before by just changing the parametrization a bit. All the generators which have an entry on the diagonal will get an additional -1 in the very first element. They now look like this:

$$T_i{}^j = -\mathrm{i}\left[e_i \otimes e^j - \delta_i^j e_1 \otimes e^1\right].$$

This makes the generator $T_1{}^1$ useless. The unit determinant gave us one constraint, so the group now has one dimension less and therefore also needs one generator less. ✓

1.c. Special orthogonal group 5/5

Special orthogonal matrices are such that $O^{-1} = O^{\mathrm{T}}$. Here the exponential map is of great help. Let the algebra element o be the generator of the group element O. Then we can rewrite the defining property.

$$\begin{aligned} O^{-1} &= O^{\mathrm{T}} \\ \exp(\mathrm{i}o)^{-1} &= \exp(\mathrm{i}o)^{\mathrm{T}} \\ \exp(-\mathrm{i}o) &= \exp(\mathrm{i}o^{\mathrm{T}}) \\ -o &= o^{\mathrm{T}} \end{aligned}$$ ✓

We now want antisymmetric generators. The group is a compact lie group because the parameters which we are going to introduce later are defined on compact support. Compact groups can have hermitian generators. This is the first compact group, so the generators actually are hermitian and purely imaginary, therefore antisymmetric.

Rotations can be parametrized by angles around all the axes in the space. One such active rotation around Θ^i can be written as such:

$$O(\Theta^i) = \begin{pmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & \cos(\Theta^i) & \cdots & -\sin(\Theta^i) & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \sin(\Theta^i) & \cdots & \cos(\Theta^i) & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 1 \end{pmatrix},$$ ✓

there are ones on the diagonal except for the elements where the actual rotation takes place. The generators of this group are then simply

$$T_{ij} = -\mathrm{i} e_{[i} \otimes e_{j]}.$$

In three dimensions, there are only three distinct rotations. In four dimensions there are already six possible rotations, which we know from the Lorentz group with its six (pseudo-)rotations. The rotation angles should be given as an antisymmetric matrix instead of a vector. We regularly use a vector in three dimensions, which is again a hidden Hodge dual nobody told us about before.

The components of the generators can be written as

$$\left[T_{ij}\right]^{ab} = -\mathrm{i}\delta^a_{[i}\delta^b_{j]}$$

or alternatively

$$\left[T_{ij}\right]_{ab} = -\mathrm{i}\left[\eta_{ai}\eta_{bj} - \eta_{aj}\eta_{bi}\right]$$

which can be generalized to any matrix group $\mathrm{SO}(n-p, p, \mathbf{R})$ if the correct signature of the metric tensor is inserted. The first version is also given similarly by Peskin and Schroeder (1995, (3.18)). ✓

1.d. Unitary group 5/5

Unitary matrices U have the property $U^\dagger = U^{-1}$. This can be used with the exponential map like with the orthogonal matrices to give a constraint for the generator u.

$$\begin{aligned} U^\dagger &= U^{-1} \\ \exp(\mathrm{i}u)^\dagger &= \exp(\mathrm{i}u)^{-1} \\ \exp(-\mathrm{i}u^\dagger) &= \exp(-\mathrm{i}u) \\ u^\dagger &= u \end{aligned}$$ ✓

So again we have have hermitian generators for this group. The group itself is not compact though, as can be seen later when parametrizing it.

We know that the Pauli matrices are one basis for the *special* unitary algebra in the two dimensional representation and the Gell-Mann matrices the basis in the three dimensional representation. Since we regard the general unitary group, we need one additional real parameters such that the determinant can have any argument, while the modulus is still 1. Adding the identity matrix (or in physicist's convention the matrix $-\mathrm{i}1_n$) to the generators will basically give us $\boxed{\mathrm{SU}(n) \times \mathrm{U}(1) = \mathrm{U}(n).}$ is that really true?

The real part of the trace of the generator does not have to be zero as we are allowing a complex phase for this group. This means that our generators fall in three categories. what about det = −1 ?

Imaginary off-diagonal There are generators which looks like these:

$$\begin{pmatrix} 0 & -\mathrm{i}\alpha & 0 \\ \mathrm{i}\alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & -\mathrm{i}\alpha \\ 0 & 0 & 0 \\ \mathrm{i}\alpha & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i}\alpha \\ 0 & \mathrm{i}\alpha & 0 \end{pmatrix}.$$

There are $n[n-1]/2$ of those. This can be seen since a $n \times n$ matrix has n^2 elements. We take away the diagonal and are at $n[n-1]$. Then only the upper or lower elements are unique, the other half is determined since the matrix is supposed to be hermitian.

Real on-diagonal Since we do not have to take care of the trace, we can add n generators of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Real off-diagonal And there are another $n[n-1]/2$ generators for the real off-diagonal entries.

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Together we have n^2 generators with real parameters. This means that the dimension of the algebra $\mathfrak{u}(n)$ has half the dimension compared to $\mathfrak{gl}(n, \mathbf{R})$. The group $\mathrm{SU}(n)$ has $n^2 - 1$ generators, which fits with the three Pauli and eight Gell-Mann matrices. As always, the algebra is the space spanned by those n^2 generators.

1.e. Lorentz group 5/5

With the introduction of the metric tensor for the special orthogonal group, we already derived this for the Lorentz group as well. Just set the metric tensor to $\eta = \mathrm{diag}(1, -1, -1, -1)$.

Using Greek indices we have the six generators

$$[T^{\mu\nu}]^{\rho\lambda} = -\mathrm{i}\left[\eta^{\rho\mu}\eta^{\lambda\nu} - \eta^{\rho\nu}\eta^{\lambda\mu}\right].$$

The corresponding angles go into an angle two-form ω. The elements ω_{0i} correspond to pseudo-rotations or boosts, whereas the elements ω_{ij} correspond to rotations about the axes k (with ϵ_{ijk} in mind).

good!

1.f. Heisenberg group 5/5

Thankfully the Heisenberg group is already given with parameters. So we can just write down the generators:

$$T_1 = -\mathrm{i}e_1 \otimes e^2, \qquad T_3 = -\mathrm{i}e_1 \otimes e^3, \qquad T_3 = -\mathrm{i}e_2 \otimes e^3$$

And then the algebra simply is

$$\mathfrak{h} = \left\{-\mathrm{i}\alpha^i T_i : \alpha^i \in \mathbf{R}, i \in \{1,2,3\}\right\}$$ ✓

2. Field matter coupling in quantum mechanics

2.a. Einstein-Lorentz force 5/5

The Lagrangian with all explicit functional dependence written down is

$$L(x,\dot{x}) = \frac{1}{2}\eta_{ij}\dot{x}^i\dot{x}^j - \phi(x,t) + \dot{x}^i A_i(x,t).$$ ✓

We use η instead of g as long as we are in coordinate frame without curvature.

The Euler-Lagrange equation for this is $\delta L = 0$ where δ means the variation. We have (✓)

↳ with respect to x^i

$$0 = \frac{\partial L}{\partial x^i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}^i}$$

We compute the partial derivatives.

$$0 = -\phi_{,i} + \dot{x}^j A_{j,i} - \frac{\mathrm{d}}{\mathrm{d}t}\left[\dot{x}_i + A_i(x,t)\right]$$

The total time derivative will lead to a chain rule in $\dot{x}$.

$$0 = -\phi_{,i} + \dot{x}^j A_{j,i} - \ddot{x}_i - A_{i,j}\dot{x}^j - \dot{A}_i$$

We move $\ddot{x}_i$ to the other side.

$$\ddot{x}_i = -\phi_{,i} + \dot{x}^j A_{j,i} - A_{i,j}\dot{x}^j - \dot{A}_i$$

With a bit of vector calculus magic we can regroup those terms. It is easiest to start with the result and check using the Levi-Civita symbols that it matches the expression in the line above.

$$\ddot{x}_i = \underbrace{-\nabla\phi}_{E} + \dot{x} \times \underbrace{[\nabla \times A]}_{B}$$

$$\ddot{x}_i = E + \dot{x} \times B$$

And that is the Einstein-Lorentz equation. ✓

2.b. Hamiltonian 5/5

The canonical momenta are

$$\pi_i = \frac{\partial L}{\partial \dot{x}^i} = \dot{x}_i + A_i(\boldsymbol{x}, t).$$

We assemble the Hamiltonian using the Legendre transformation.

$$\begin{aligned} H &= \pi_i \dot{x}^i - L \\ &= \pi_i \dot{x}^i - \frac{1}{2}\dot{x}_i \dot{x}^i - \phi(\boldsymbol{x}, t) + \dot{x}^i A_i(\boldsymbol{x}, t) \\ &= \pi_i \left[\pi^i - A^i(\boldsymbol{x}, t)\right] - \frac{1}{2}[\pi_i - A_i(\boldsymbol{x}, t)]\left[\pi^i - A^i(\boldsymbol{x}, t)\right] + \phi(\boldsymbol{x}, t) - \left[\pi^i - A^i(\boldsymbol{x}, t)\right] A_i(\boldsymbol{x}, t) \end{aligned}$$

We can factor the first and last term together and have two times the second term. Together we only have the positive second term left.

$$= \frac{1}{2}[\pi_i - A_i]\left[\pi^i - A^i\right] + \phi$$

2.c. Gauge transformation 4/5

The time dependant Schrödinger equation is

$$\mathrm{i}\frac{\partial}{\partial t}\psi(\boldsymbol{x}, t) = H(\boldsymbol{x}, t)\psi(\boldsymbol{x}, t).$$

Inserting our expression from the previous part and replacing π_i with $-\mathrm{i}\frac{\partial}{\partial i}$ we arrive at the given Schrödinger equation.

We start with the transformation of the magnetic potential.

$$\mathrm{i}\dot{\psi} = \left[\frac{1}{2}\left[-\mathrm{i}\partial_i - A'_i + \theta_{,i}\right]\left[-\mathrm{i}\partial^i - A^{i\prime} + \theta^{,i}\right] + \phi' + \dot{\theta}\right]\psi(\boldsymbol{x}, t)$$

The terms with θ get factored out.

$$= \left[\frac{1}{2}\left[\left[-\mathrm{i}\partial_i - A'_i\right]\left[-\mathrm{i}\partial^i - A^{i\prime}\right] + \left[-\mathrm{i}\partial_i - A'_i\right]\theta^{,i} + \theta_{,i}\left[-\mathrm{i}\partial^i - A^{i\prime}\right] + \theta_{,i}\theta^{,i}\right] + \phi' + \dot{\theta}\right]\psi$$

We extract the original Hamiltonian.

$$= \left[H' + \frac{1}{2}\left[\left[-\mathrm{i}\partial_i - A'_i\right]\theta^{,i} + \theta_{,i}\left[-\mathrm{i}\partial^i - A^{i\prime}\right] + \theta_{,i}\theta^{,i}\right] + \dot{\theta}\right]\psi$$

The remaining inner brackets are expanded, keeping the product rule in mind.

$$= \left[H' + \frac{1}{2}\left[-\mathrm{i}\theta^{,i}{}_{,i} - A'_i\theta^{,i} - \mathrm{i}\theta^{,i}\partial_i - \mathrm{i}\theta_{,i}\partial^i - \theta_{,i}A^{i\prime} + \theta_{,i}\theta^{,i}\right] + \dot{\theta}\right]\psi$$

Two terms occur twice, we group them.

$$= \left[H' + \frac{1}{2} \left[-\mathrm{i}\theta^{,i}{}_{,i} - 2A'_i \theta^{,i} - 2\mathrm{i}\theta^{,i}\partial_i + \theta_{,i}\theta^{,i} \right] + \dot{\theta} \right] \psi$$

There are terms left that we cannot get rid of. They prevent a form-invariance of the Schrödinger equation. ✓

yes ✓

The solution is to add the phase transformation to the wave function. So we replace ψ by $\exp(-\mathrm{i}\theta)\psi'$. First we compute the commutator of H' with $\exp(-\mathrm{i}\theta)$.

$$\left[H', \exp(-\mathrm{i}\theta)\right]\psi = \left[\left[\frac{1}{2} \left[-\mathrm{i}\partial_i - A'_i \right] \left[-\mathrm{i}\partial^i - A'^i \right] + \phi' \right], \exp(-\mathrm{i}\theta) \right] \psi$$

Every summand which does not contain a differential operator will not contribute to the commutator. ✓ We can therefore already eliminate ϕ'.

$$= \frac{1}{2} \left[\left[-\mathrm{i}\partial_i - A'_i \right] \left[-\mathrm{i}\partial^i - A'^i \right], \exp(-\mathrm{i}\theta) \right] \psi$$

Then we expand the brackets.

$$= \frac{1}{2} \left[-\partial_i \partial^i - \mathrm{i}\partial_i A'^i - \mathrm{i}A'_i \partial^i + A'_i A^{i\prime}, \exp(-\mathrm{i}\theta) \right] \psi$$

We apply the product rule in the second term.

$$= \frac{1}{2} \left[-\partial_i \partial^i - \mathrm{i}A'^i{}_{,i} - \mathrm{i}A^{i\prime}\partial_i - \mathrm{i}A'_i \partial^i + A'_i A^{i\prime}, \exp(-\mathrm{i}\theta) \right] \psi$$

Again, everything without differential operators does not contribute.

$$= \frac{1}{2} \left[-\partial_i \partial^i - 2\mathrm{i}A'_i \partial^i, \exp(-\mathrm{i}\theta) \right] \psi$$

We write out the commutator in full.

$$= -\frac{1}{2} \left[\left[\partial_i \partial^i + 2\mathrm{i}A'_i \partial^i \right] \exp(-\mathrm{i}\theta)\psi - \exp(-\mathrm{i}\theta) \left[\partial_i \partial^i + 2\mathrm{i}A'_i \partial^i \right] \psi \right]$$

We apply the product rule for the first partial derivative.

$$= -\frac{1}{2} \left[\left[-\mathrm{i}\partial_i \exp(-\mathrm{i}\theta)\theta^{,i}\psi + \partial_i \exp(-\mathrm{i}\theta)\partial^i\psi - \mathrm{i}2A'_i \exp(-\mathrm{i}\theta)\theta^{,i}\psi + 2\mathrm{i}A'_i \exp(-\mathrm{i}\theta)\partial^i\psi \right] \right.$$
$$\left. - \exp(-\mathrm{i}\theta) \left[\partial_i \partial^i + 2\mathrm{i}A'_i \partial^i \right] \psi \right]$$

One term can already be cancelled.

$$
\begin{aligned}
&= -\frac{1}{2}\left[\left[-\mathrm{i}\partial_i \exp(-\mathrm{i}\theta)\theta^{,i}\psi + \partial_i \exp(-\mathrm{i}\theta)\partial^i\psi - \mathrm{i}2A'_i \exp(-\mathrm{i}\theta)\theta^{,i}\psi\right] - \exp(-\mathrm{i}\theta)\partial_i\partial^i\psi\right] \\
&= -\frac{1}{2}\exp(-\mathrm{i}\theta)\left[-\theta_{,i}\theta^{,i} - \mathrm{i}\theta^{,i}{}_{,i} - \mathrm{i}\theta^{,i}\partial_i + \mathrm{i}\theta_{,i}\partial^i + \partial_i\partial^i - 2\mathrm{i}A'_i\theta^{,i} - \partial_i\partial^i\right]\psi
\end{aligned}
$$

We can cancel several more terms here.

$$
= -\frac{1}{2}\exp(-\mathrm{i}\theta)\left[-\theta_{,i}\theta^{,i} - \mathrm{i}\theta^{,i}{}_{,i} - 2\mathrm{i}A'_i\theta^{,i}\right]\psi
$$

These terms almost match up with the extra terms we have in the Schrödinger equation. The time derivative on the other side will give us the $-\dot{\theta}$ term that we also need to cancel it in the Schrödinger equation. ✓

→ show that explicitly!

2.d. Comparison 3/5

From what we have heard before this lecture, we would identify:

- Bundle of vector spaces: Space of L^2 functions for ψ over $\mathbf{R}^3$ actually, here it is just $\mathbb{C}$
- Gauge group: $\mathrm{U}(1) \simeq \mathrm{S}^1$ ✓
- Lie algebra: $\mathfrak{u}(1) \simeq \mathbf{R}$ ✓
- Matter field: Particle wavefunction $\psi(\boldsymbol{x}, t)$ ✓ good!
- Connection: Magnetic potential $\boldsymbol{A}$

? electromagnetic potential $(\phi, A) \mathrel{\hat{=}} A_\mu$

References

Peskin, Michael E. and Daniel V. Schroeder (1995). *An Introduction to Quantum Field Theory*. Westview Press. ISBN: 978-0-201-50397-5.