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Geometry in Physics

Problem Set 10

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problem	achieved points	possible points
Lie algebra	10	10
Exponential map	18	20
Lie algebra 2	~~20~~ 18	20
Total	46	50

1. Lie algebra

1.a. Matrix commutator 5/5

Linearity The matrix group $\mathrm{GL}(n, \mathbf{R})$ where we get our matrices from is a vector space. The commutator of matrices therefore is linear in the arguments:

$$[\lambda A + \mu B, C] = [\lambda A + \mu B]C - C[\lambda A + \mu B]$$

The matrix multiplication is associative and also has the distributive laws.

$$= \lambda AC + \mu BC - C\mu B - C\lambda A$$

Then we move the factors up front and reorder the summands.

$$= \lambda AC - \lambda CA + \mu BC - \mu CB$$

We can factor out the factors and get recognize the matrix commutators again.

$$= \lambda[A, C] + \mu[B, C] \quad ✓$$

The linearity in the second argument follows from the anti-commutativity of the matrix commutator.

Anti-commutativity The way the commutator is defined it is easy to see that it is anti-symmetric in its arguments. ✓

Jacobi identity We expand the inner commutator.

$$[A,[B,C]]+[C,[A,B]]+[B,[C,A]]=[A,BC-CB]+[C,AB-BA]+[B,CA-AC]$$

We expand the outer commutator.

$$=ABC-ACB-BCA+CBA+CAB-CBA-ABC+BAC+BCA-BAC-CAB+ACB$$

Then we sort the terms.

$$=ABC-ABC+ACB-ACB+BAC-BAC+BCA-BCA+CAB-CAB+CBA-CBA$$

And they all cancel each other.

$$=0$$ ✓

1.b. Cross product 5/5

Linearity Using components one can lead this back to the linearity of a bilinear form:

$$\boldsymbol{a}\times\boldsymbol{b}=\epsilon_{ijk}a^i b^j \boldsymbol{e}^k.$$

This expression is linear in a^i and b^j each.

Anti-commutativity If we exchange $\boldsymbol{a}$ and $\boldsymbol{b}$ in the above expression, we have to exchange the indices in the Levi-Civita symbol and therefore get a minus sign.

$$\boldsymbol{a}\times\boldsymbol{b}=\epsilon_{ijk}a^i b^j \boldsymbol{e}^k$$

We exchange the indices of the Levi-Civita symbol.

$$=-\epsilon_{jik}a^i b^j \boldsymbol{e}^k$$

For consistency, we can also exchange the components of $\boldsymbol{a}$ and $\boldsymbol{b}$ here. Those are just regular numbers and therefore do not cause a change in sign.

$$=-\epsilon_{jik}b^j a^i \boldsymbol{e}^k$$

By the definition, this is a cross product in the other order.

$$=-\boldsymbol{b}\times\boldsymbol{a}$$ ✓

Jacobi identity We will need to write it out again here. This time we get to do some gymnastics with Levi-Civita symbols. We first expand everything in terms of components.

$$\begin{aligned} &\boldsymbol{a} \times [\boldsymbol{b} \times \boldsymbol{c}] + \boldsymbol{c} \times [\boldsymbol{a} \times \boldsymbol{b}] + \boldsymbol{b} \times [\boldsymbol{c} \times \boldsymbol{a}] \\ &= \epsilon_{ijk} a^i \epsilon^j{}_{lm} b^l c^m \boldsymbol{e}^k + \epsilon_{ijk} c^i \epsilon^j{}_{lm} a^l b^m \boldsymbol{e}^k + \epsilon_{ijk} b^i \epsilon^j{}_{lm} c^l a^m \boldsymbol{e}^k \end{aligned}$$

We can factor out the Levi-Civita symbols and the basis vector.

$$= \epsilon_{ijk} \epsilon^j{}_{lm} \left[a^i b^l c^m + c^i a^l b^m + b^i c^l a^m \right] \boldsymbol{e}^k$$

Then we can contract both symbols and obtain some metric tensors.

$$= [g_{kl} g_{im} - g_{km} g_{li}] \left[a^i b^l c^m + c^i a^l b^m + b^i c^l a^m \right] \boldsymbol{e}^k$$

✓

We apply those metric tensors and lower some indices with them.

$$= \left[a^i b^k c_i + c^i a^k b_i + b^i c^k a_i - a^i b_i c^k - c^i a_i b^k - b^i c_i a^k \right] \boldsymbol{e}_k$$

The terms need to be reordered.

$$= \left[a^i b^k c_i - a_i b^k c^i + a^k b_i c^i - a^k b^i c_i + a_i b^i c^k - a^i b_i c^k \right] \boldsymbol{e}_k$$

And now the terms cancel and we get zero.

$$= 0$$

✓ nicely done!

2. Exponential map

(a) Using the chain rule and using that a matrix will commute with a smooth function of itself, the desired property

$$\frac{\mathrm{d}}{\mathrm{d}t} \exp(tA) = A \exp(tA)$$

directly follows. But one can use the definition to show this as well, of course:

$$\frac{\mathrm{d}}{\mathrm{d}t} \exp(tA) = \frac{\mathrm{d}}{\mathrm{d}t} \sum_{k=0}^{\infty} \frac{[tA]^k}{k!}$$

The differentiation with respect to t is a limit, the sum is also a limit. Since the exponential function is one of the smoothest and most differentiable functions known to Physicists, it should be possible to interchange those two limits. ✓

$$= \sum_{k=0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}t} \frac{t^k A^k}{k!}$$

✓

We apply the derivative. The case $k = 0$ will give zero, so we exclude that from the summation.

$$= \sum_{k=1}^{\infty} \frac{t^{k-1} A^k}{[k-1]!}$$

We can move one A to the front and relabel the summation index $n = k - 1$.

$$= A \sum_{n=0}^{\infty} \frac{t^n A^n}{n!}$$

Now we can use the definition of the exponential function again and wrap this up.

$$= A \exp(tA)$$

(b) We have $AB = BA$, they commute. This is our first approach, but we got stuck a bit later. To show all the steps, we again start by expanding the exponential functions.

$$\exp(A + B) = \sum_{k=0}^{\infty} \frac{[A+B]^k}{k!}$$

Now we need to use the binomial theorem to split this sum. Since A and B commute, we can actually do this and do not have to carry 2^k individual terms.

$$= \sum_{k=0}^{\infty} \sum_{p=0}^{k} \binom{k}{p} \frac{A^p B^{k-p}}{k!}$$

We expand the binomial coefficient.

$$= \sum_{k=0}^{\infty} \sum_{p=0}^{k} \frac{k!}{[k-p]! p!} \frac{A^p B^{k-p}}{k!}$$

Now we can cancel the factorial from the original exponential series. Then we can write this in a suggestive way.

$$= \sum_{k=0}^{\infty} \sum_{p=0}^{k} \frac{A^p}{p!} \frac{B^{k-p}}{[k-p]!}$$

This is our weakest part of our derivation. We would argue that by this double sum all powers of A and B occur exactly once with their corresponding factorial. Therefore we can reorder the summations completely and just write this with new indices as

$$= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{A^k}{k!} \frac{B^l}{l!}.$$

Then we can factor by moving the summation sign.

$$= \sum_{k=0}^{\infty} \frac{A^k}{k!} \sum_{l=0}^{\infty} \frac{B^l}{l!}$$

And this is the definition again and we get our final result.

$$= \exp(A)\exp(B)$$

There is another neat derivation which is based on a trick we had to find first. Using the result of (a) we can start with an auxiliary function F.

$$F(t) := \exp([A+B]t) - \exp(At)\exp(Bt)$$

We differentiate with respect to t which is usable as of part (a). Then we obtain:

$$\dot{F}(t) = [A+B]\exp([A+B]t) - A\exp(At)\exp(Bt) - \exp(At)B\exp(Bt)$$

We can factor the exponentials in the last two terms and combine that with the very first term.

$$= [A+B][\exp([A+B]t) - \exp(At)\exp(Bt)]$$

And then we can recognize the function F again.

$$= [A+B]F(t)$$

The only sensible solution to this differential equation is $F(t) = 0$. This then implies that

$$\exp([A+B]t) = \exp(At)\exp(Bt)$$

If A and B do not commute, this relation does not hold since one could not apply the binomial theorem. There will be a third exponential with the commutators and higher order commutators. That is the Baker-Campbell-Hausdorff formula.

(c) Use the previous result with $A = sA$ and $B = tA$.

(d) We have a matrix from $\mathrm{GL}(n, \mathbf{R})$, so we cannot say that it can be diagonalized. There always exists the Jordan form which has the eigenvalues on the diagonal. Since the result will not differ, we will assume that A is diagonalizable. Then we can find an eigenvalue matrix Λ and a transform S such that $A = S\Lambda S^{-1}$.

$$\det(\exp(A)) = \det\left(\exp(S^{-1}\Lambda S)\right)$$

you have to show that they won't differ!

Using part (g) of this problem, we can pull out S.

$$= \det\left(S^{-1}\exp(\Lambda)S\right)$$

The determinant of a product is the product of the determinants.

$$= \det \boldsymbol{S}^{-1} \det(\exp(\boldsymbol{\Lambda})) \det \boldsymbol{S}$$

The determinants of the transform will just cancel.

$$= \det(\exp(\boldsymbol{\Lambda}))$$

The eigenvalue matrix diagonal, so we just have a matrix with the exponentiated eigenvalues on the diagonal. The determinant of such a diagonal matrix is just the product of all the elements on the diagonal.

$$= \prod_i \exp(\lambda_i)$$

We can combine those exponentials.

$$= \exp(\sum_i \lambda_i)$$

And that is the trace of the matrix $\boldsymbol{\Lambda}$.

$$= \exp(\operatorname{tr} \boldsymbol{\Lambda})$$

Now we add the $\boldsymbol{S}$ again.

$$= \exp\left(\operatorname{tr}(\boldsymbol{S}\boldsymbol{S}^{-1}\boldsymbol{\Lambda})\right)$$

The trace is cyclic, so we can move the $\boldsymbol{S}$ to the other end.

$$= \exp\left(\operatorname{tr}(\boldsymbol{S}^{-1}\boldsymbol{\Lambda}\boldsymbol{S})\right)$$

This expression is $\boldsymbol{A}$.

$$= \exp(\operatorname{tr} \boldsymbol{A})$$

For the general case one would have to use the Jordan form and its powers, but the idea is the same.

(e) Using the previous result, having $\det(\exp(\boldsymbol{A})) = 0$ would mean that $\exp(\operatorname{tr} \boldsymbol{A}) = 0$. However, the real matrix cannot have a trace such that the exponential of it would be zero. This would not even work with a complex matrix.

(f) We start with the definition of the inverse in general.

$$\exp(\boldsymbol{A})\exp(\boldsymbol{A})^{-1} = \mathbf{1}$$

Then we write the unit matrix as an exponentiated zero matrix. With the exponential map of the Lie algebra in mind this just means that we take an arbitrary generator and set the parameter vector to zero.

$$\exp(A)\exp(A)^{-1} = \exp(0)$$

Now we just use the definition of the additive inverse.

$$\exp(A)\exp(A)^{-1} = \exp(A - A)$$

As we have shown in (a), we can split the sum inside the exponential into the product of two exponentials.

$$\exp(A)\exp(A)^{-1} = \exp(A)\exp(-A) \quad ✓$$

And then we cancel the first exponential and get the desired expression.

$$\exp(A)^{-1} = \exp(-A) \quad ✓$$

(g) We can move the S and S^{-1} into the power since the $S^{-1}S$ pairs in between the A will just cancel. See $[SAS^{-1}]^2 = SAS^{-1}SAS^{-1} = SA^2S^{-1}$.

$$S\exp(A)S^{-1} = \sum_{k=0}^{\infty} S\frac{A^k}{k!}S^{-1} = \sum_{k=0}^{\infty} \frac{[SAS^{-1}]^k}{k!} = \exp(SAS^{-1})$$

This would break if S would be singular, so we needed to have $\det(S) \neq 0$. ✓

(h) The transpose changes the order in a product, but that does not hurt since we only have powers of A here:

$$\exp(A^{\mathrm{T}}) = \left[\sum_{k=0}^{\infty} \frac{[A^{\mathrm{T}}]^k}{k!}\right] \quad ✓$$

We can move the transpose outside of the bracket since the reversal in the order of the k factors of A does not change anything.

$$= \left[\sum_{k=0}^{\infty} \frac{[A^k]^{\mathrm{T}}}{k!}\right]$$

We can then use the linearity of the transpose and move it out of the sum.

$$= \left[\sum_{k=0}^{\infty} \frac{A^k}{k!}\right]^{\mathrm{T}}$$

And then we recover the other side of the desired equation.

$$= [\exp(A)]^{\mathrm{T}} \quad ✓$$

(i) This is very similar to (d). We can just diagonalize the matrix A in the exponential and have a diagonal eigenvalue matrix Λ, where $\exp(\Lambda)$ then contains the exponentiated eigenvalues of A. And any matrix can be diagonalized over the complex numbers. ✓

3. Lie algebra 2

3.a. Basis 3/5

The matrices S_i are the generators, but in the mathematician's convention of anti-hermitian generators. This means that there are no additional imaginary units are build in everywhere. ✓

There are three parameters here: a, $\operatorname{Re} z$ and $\operatorname{Im} z$ or equivalently a, z and $\bar{z}$. To get the generators we want we choose the first parametrization of the Lie algebra. Then we can compute the generators for the general $\mathfrak{su}(2)$ algebra element $s(a, \operatorname{Re} z, \operatorname{Im} z)$:

$$S_i = \frac{\mathrm{d}s}{\mathrm{d}\alpha^i}(0)$$

where $\boldsymbol{\alpha}$ is the parameter vector. The physicist's convention would add $-\mathrm{i}$ to this equation. The general element is written

$$s(a, \operatorname{Re} z, \operatorname{Im} z) = \begin{pmatrix} \mathrm{i}a & -\operatorname{Re} z + \mathrm{i} \operatorname{Im} z \\ \operatorname{Re} z + \mathrm{i} \operatorname{Im} z & -\mathrm{i}a \end{pmatrix}.$$ ✓

Taking those derivatives is straightforward. The additional factor of one half is just a convention that will make the structure constants in (b) nicer and will set the scaling factor in the scalar product for generators. We are not going to show the three matrix derivatives here explicitly since they now should be straightforward to read off.

show linear independence!

3.b. Lie product 10/10

We have to derive the structure constants here. As physicists we see their relation to the Pauli matrices as $S_i = -\frac{1}{2}\mathrm{i}\sigma_i$. We know the structure constants for the Pauli matrix algebra, which are $2\mathrm{i}\epsilon_{ij}{}^k$. Inserting all this gives Equation (11) from the problem set. ✓

3.c. Universal covering group

The exponentiation of $\mathfrak{su}(2)$ should yield the universal covering group, which is isomorphic to SU(2) and $\mathrm{SO}(3) \times \mathbf{Z}_2$. The general element from the universal covering group looks like this in the mathematician's convention:

$$B(\boldsymbol{\alpha}) = \exp(\alpha^i S_i).$$

The generators S_i have no trace, which means that the matrices $\boldsymbol{B}$ we get from the exponentiation will have unit determinant. We therefore will have some special matrix group in the end. ✓

So let us do this calculation. One can approach this with an Euler angle prescription where we do not exponentiate the full expression $\alpha^i S_i$ at once, but exponentiate each generate on its own and matrix multiply the results.

$$\begin{aligned} B(\alpha) &= \prod_{i=1}^{3} \exp(\alpha^i S_i) \qquad \text{no summation convention} \\ &= \exp(\alpha^1 S_1)\exp(\alpha^2 S_2)\exp(\alpha^3 S_3) \end{aligned}$$

All those matrix exponentials work in the same way. The generator squared gives something proportional to the negative unit matrix. Therefore the terms in the power series of the exponential fall into even and odd terms and can then be grouped using sine and cosine functions, see below for the detailed steps. The last one is easy since it is a diagonal matrix. We obtain with a help from Mathematica

$$= \begin{pmatrix} \cos\left(\frac{\alpha_1}{2}\right) & \mathrm{i}\sin\left(\frac{\alpha_1}{2}\right) \\ \mathrm{i}\sin\left(\frac{\alpha_1}{2}\right) & \cos\left(\frac{\alpha_1}{2}\right) \end{pmatrix} \begin{pmatrix} \cos\left(\frac{\alpha_2}{2}\right) & -\sin\left(\frac{\alpha_2}{2}\right) \\ \sin\left(\frac{\alpha_2}{2}\right) & \cos\left(\frac{\alpha_2}{2}\right) \end{pmatrix} \begin{pmatrix} \exp\left(\mathrm{i}\frac{\alpha_3}{2}\right) & 0 \\ 0 & \exp\left(\mathrm{i}\frac{\alpha_3}{2}\right) \end{pmatrix}.$$ (✓)

Then we build up the product of those three matrices. The matrix is rather big, so we list it here in components:

$$\begin{aligned} B_{11} &= \exp\left(\mathrm{i}\frac{\alpha_3}{2}\right)\left[\cos\left(\frac{\alpha_1}{2}\right)\cos\left(\frac{\alpha_2}{2}\right) + \mathrm{i}\sin\left(\frac{\alpha_1}{2}\right)\sin\left(\frac{\alpha_2}{2}\right)\right] \\ B_{21} &= \left[\frac{1}{2} + \frac{\mathrm{i}}{2}\right]\exp\left(-\mathrm{i}\frac{\alpha_3}{2}\right)\left[\sin\left(\frac{\alpha_1 - \alpha_2}{2}\right) + \mathrm{i}\sin\left(\frac{\alpha_1 + \alpha_2}{2}\right)\right] \\ B_{12} &= \exp\left(\mathrm{i}\frac{\alpha_3}{2}\right)\left[\mathrm{i}\sin\left(\frac{\alpha_1}{2}\right)\cos\left(\frac{\alpha_2}{2}\right) + \cos\left(\frac{\alpha_1}{2}\right)\sin\left(\frac{\alpha_2}{2}\right)\right] \\ B_{22} &= \exp\left(-\mathrm{i}\frac{\alpha_3}{2}\right)\left[\cos\left(\frac{\alpha_1}{2}\right)\cos\left(\frac{\alpha_2}{2}\right) - \mathrm{i}\sin\left(\frac{\alpha_1}{2}\right)\sin\left(\frac{\alpha_2}{2}\right)\right] \end{aligned}$$

One can check that this matrix is unitary, so we can conclude that this matrix comes from SU(2). ✓

Matrix exponentiation by hand And to show that we only used Mathematica for convenience and that we could have done it by hand as well, we will show the exponentiation of S_1 here in detail.

$$\exp(\alpha^1 S_1) = \exp\left(\frac{\alpha^1}{2}\begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}\right)$$

It becomes easier when the matrix is simpler.

$$= \exp\left(\frac{\mathrm{i}\alpha^1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$$

We expand the exponential into a power series.

$$= \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{\mathrm{i}\alpha^1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]^k$$

Now the square of the matrix is the identity. We can therefore group this into two summands.

$$= \sum_{k \text{ even}} \frac{1}{k!} \left[\frac{\mathrm{i}\alpha^1}{2}\right]^k \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sum_{k \text{ odd}} \frac{1}{k!} \left[\frac{\mathrm{i}\alpha^1}{2}\right]^k \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Then we change the index and generate even and odd numbers by hand.

$$= \sum_{n=0}^{\infty} \frac{1}{[2n]!} \left[\frac{\mathrm{i}\alpha^1}{2}\right]^{2n} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sum_{n=0}^{\infty} \frac{1}{[2n+1]!} \left[\frac{\mathrm{i}\alpha^1}{2}\right]^{2n+1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The imaginary unit can be split off and take the factor two with it.

$$= \sum_{n=0}^{\infty} \frac{[-1]^n}{[2n]!} \left[\frac{\alpha^1}{2}\right]^{2n} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \mathrm{i} \sum_{n=0}^{\infty} \frac{[-1]^n}{[2n+1]!} \left[\frac{\alpha^1}{2}\right]^{2n+1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Those are the power series versions of cosine and sine, respectively.

$$= \cos\left(\frac{\alpha^1}{2}\right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \mathrm{i} \sin\left(\frac{\alpha^1}{2}\right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

We can then combine the two matrices and obtain the same result we got with Mathematica.

$$= \begin{pmatrix} \cos\left(\frac{\alpha^1}{2}\right) & \mathrm{i}\sin\left(\frac{\alpha^1}{2}\right) \\ \mathrm{i}\sin\left(\frac{\alpha^1}{2}\right) & \cos\left(\frac{\alpha^1}{2}\right) \end{pmatrix}$$