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This is a problem set (as turned in) for the module geometry.

This problem set is not reviewed by a tutor. This is just what I have turned in.

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[disclaimer]

Geometry in Physics

Problem Set 8

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problem	achieved points	possible points
Poynting's theorem		20
Transformation properties of the metric		10
Lorentz group 1		5
Lorentz group 2		10
Total		45

1 Poynting's theorem

1. The field D is a two-form but only has one index. In this case it is the Hodge dual of the real thing and should have an upper index.

$$\begin{aligned} W_\mathrm{e} &= \frac{1}{2} E \wedge D = \frac{1}{2} E_l \, \mathrm{d}x^l \wedge \epsilon_{ijk} D^k \, \mathrm{d}x^i \wedge \mathrm{d}x^j = \frac{1}{2} E_l D^k \epsilon_{ijk} \, \mathrm{d}x^l \wedge \mathrm{d}x^i \wedge \mathrm{d}x^j \\ &= \frac{1}{2} E_l D^k \epsilon_{ijk} \epsilon^{lij} \, \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = E_l D^l \, \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \end{aligned}$$

   And similarly we have $W_\mathrm{m} = H_l B^l \, \mathrm{d}V$.

2. Ampère's law is $\mathbf{curl}\, \boldsymbol{H} - \dot{\boldsymbol{D}} = \boldsymbol{J}_\mathrm{f}$. In components this can be written as $\epsilon^{ijk} H_{k,j} - \dot{D}^i = J^i$. Faraday's law is $\mathbf{curl}\, \boldsymbol{E} + \dot{\boldsymbol{B}} = \boldsymbol{0}$, which is $\epsilon^{ijk} E_{k,j} + \dot{B}^i = 0$ in components. The given wedge product simply is $E \wedge H = E_i H_k \, \mathrm{d}x^i \wedge \mathrm{d}x^k$.

$$\mathrm{d}(E \wedge H) = \partial_\mu (E_i H_k) \, \mathrm{d}x^\mu \wedge \mathrm{d}x^i \wedge \mathrm{d}x^k = \left[E_{i,\mu} H_k + E_i H_{k,\mu} \right] \mathrm{d}x^\mu \wedge \mathrm{d}x^i \wedge \mathrm{d}x^k$$

   We can split off the time derivative.

$$\begin{aligned} &= \left[\dot{E}_i H_k + E_i \dot{H}_k \right] \mathrm{d}t \wedge \mathrm{d}x^i \wedge \mathrm{d}x^k + \left[E_{i,j} H_k + E_i H_{k,j} \right] \epsilon^{jik} \, \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \\ &= \left[\dot{E}_i H_k + E_i \dot{H}_k \right] \mathrm{d}t \wedge \mathrm{d}x^i \wedge \mathrm{d}x^k + \left[H_i \dot{B}^i + E_i \dot{D}^i \right] \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \end{aligned}$$

3. $\mathrm{d}\sigma$ is the time derivative of the field energy. It is a three-form, which means that it takes a time-like and two space-like vectors. It corresponds to the Poynting vector, which can be thought of as a Hodge dual of a three-form in four-space. σ itself is an energy-density two-form. It cannot be the energy momentum tensor since that is symmetric.

4. The absence of charges means that all of Maxwell's equations are homogeneous. Then, in the sense of vectors, $\boldsymbol{E} \propto \boldsymbol{D}$ and $\boldsymbol{H} \propto \boldsymbol{B}$ which means that the wedge products are just the scalar product of the components. Depending on the choice of unit system they are even equal to each other, respectively.

5. We start with Equation (4): $\mathrm{d}\sigma = -\frac{1}{2}\partial_0[E \wedge D + H \wedge B]$. As mentioned before, in the absence of charge means that we only have simple scalar products: $\mathrm{d}\sigma = -\frac{1}{2}\partial_0[\boldsymbol{E}^2 + \boldsymbol{B}^2]\,\mathrm{d}V$. Now one integrates over the volume V and applies Stokes's theorem to the left hand side. That's it.

2 Transformation properties of the metric

1. A (mathematician's[1]) metric, which is a bilinear mapping of two vectors, has to be symmetric by definition. The matrix $\boldsymbol{g}$ which is defined by $g_{ij} := g(\boldsymbol{e}_i, \boldsymbol{e}_j)$ is therefore symmetric in its indices.

2. The metric is defined to be non-degenerate. This prohibits an eigenvalue of 0, which in turn means that $\det(\boldsymbol{g} - 0 \cdot \mathbf{1}_2)$ cannot be zero.

3. A change of basis has to leave the norm invariant. We transform the basis vectors as given: $\boldsymbol{e}'_i = \boldsymbol{e}_j [\boldsymbol{A}^{-1}]^j{}_i$. Then the components of vectors have to transform the other way around, such that the vectors themselves are not altered. This means $v^{i'} = v^j A^i{}_j$. The vector is unaltered. The transformed vector is written in components: $\boldsymbol{v}' = v^{i'} \boldsymbol{e}'_i$. Expanded, this is $v^j \boldsymbol{e}_k A^i{}_j [\boldsymbol{A}^{-1}]^k{}_i$. The inverse of the matrix time the matrix is the identity, which in components is just the Kronecker symbol. We therefore get $v^j \boldsymbol{e}_j$, which is the same vector as before. Extending this argument to tensors which are two times covariant, we need to apply the inverse transformation twice in order to preserve the action on two vectors. Writing this without indices gives the expression Equation (6) on the problem set, although one should not use the indices (ij) twice.

3 Lorentz group 1

We have R from the three dimensional "natural" representation Γ_3 of SO(3). Those transformations leave the metric $\mathbf{1}_3$ invariant. The group SO(3) also has the trivial Γ_1 representation where $\forall g \in \mathrm{SO}(3)$: $D^{\Gamma_1}(g) = 1$. We can now build a reducible representation using this: $D^{\mathrm{R}} := D^{\Gamma_1} \oplus D^{\Gamma_3}$. In this representation, we get this $\tilde{R}$. This still is an element of SO(3), albeit in a four dimensional representation.

The proper orthochronous Lorentz group contains all the elements which preserve the orientation of time and the orientation of space. The orientation of space is conserved by R since it is chosen from the *special* orthogonal group. The conservation of time requires two steps. It is decoupled from

[1] At times it becomes a bit mixed up since mathematicians call this bilinear form "metric", whereas physicists often call the metric tensor $\boldsymbol{G}$ which goes into the scalar product "metric": $\langle u, v\rangle := \boldsymbol{u}^{\mathrm{T}}\boldsymbol{G}\boldsymbol{v}$. "Metric tensor" would be a better name for this $\boldsymbol{G}$.

space since we have a reducible representation which is built up such that time and space remain invariant subspaces in the space on which the representation acts on. The representation of the group element in the temporal part is given using the trivial representation, it therefore does not change anything.

4 Lorentz group 2

1. We have given $\boldsymbol{A} \in L$ and we may assume that $\boldsymbol{A}\boldsymbol{e}_0$ is a future directed time like vector. This also means that $\eta(\boldsymbol{e}_0, \boldsymbol{A}\boldsymbol{e}_0) > 0$, which implies that the component of $A^0{}_0$ is positive. This component decides the orientation of time. Since it is positive, we have an orthochronous Lorentz transformation.

2. We have to show that the set of orthochronous Lorentz transformations leaves light like vectors as such. There is no way that any Lorentz transformation could transform a time or space like vector into a light like vector. This would require a change in norm from non-zero to zero, which is not possible with an orthogonal transformation. By the same argument, it is not possible to transform vectors from space to time like and vice versa. A time like vector can only be transformed into another such vector, although time reversal is possible by a *general* Lorentz transformation.

   The orthochronous Lorentz group has the distinguishing property that its elements do not change the time direction when applied to vectors from Minkowski space. We are interested in time like vectors to begin with, so the set Z^+ is the right choice. Since $\boldsymbol{A} \in L$ must not change the metric, the condition $\eta(\boldsymbol{A}\boldsymbol{e}_0, \boldsymbol{A}\boldsymbol{e}_0) > 0$ directly follows from $\eta(\boldsymbol{e}_0, \boldsymbol{e}_0) > 0$. The condition $\eta(\boldsymbol{e}_0, \boldsymbol{A}\boldsymbol{e}_0) = A^0{}_0 > 0$ that we impose by requiring that $\boldsymbol{A}$ maps Z^+ to itself ensures that we do not have a time reversal. Therefore this set is the orthochronous Lorentz group.