Disclaimer

This is a reviewed problem set for the module geometry.

This problem set was reviewed by a tutor. *This does not mean that it is a perfect solution. Neither I or the tutor imply that there are no further mistakes in this document.*

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[disclaimer]

Geometry in Physics

Problem Set 7

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Group 1 – Jens Boos

problem	achieved points	possible points
Maxwell's field equations	16	24
Cohomology groups of a torus	12	16
Homotopy type of a punctured plane	10	10
Total	38	50

We will choose coordinates (x, y, z) instead of the generic (x^1, x^2, x^3) since it is less typing and you liked it on sheet 5. When we need to use indices, we will still use x^i and mean the same coordinates. thank you! Electromagnetism feels to be fundamentally connected to special relativity and therefore spacetime. We hence use $ct = x^0$ and Greek indices where it eases notation. This choice of x^0 also implies that $\partial_0 = \frac{1}{c}\frac{\partial}{\partial t}$ by virtue of the chain rule.

not really, since electrodynamics can be formulated without the explicit appearance of the metric!

1. Maxwell's field equations

The B_i seems to be somewhat strange. It suggests that B is a 1-form, which it is not. We are fine with B_{ij} and B^k, ~~where the latter is the Hodge dual of the 2-form.~~ We will therefore write B^i here. ok.

no! $\star B$ is a 1-form not a vector!

1.a. Faraday's lay of induction 6/6

In order so make this shorter, we rewrite the definition of the 2-form α:

$$\alpha = E_i \,\mathrm{d}x^i \wedge \mathrm{d}x^0 + \frac{1}{2}\epsilon_{ijk} B^i \,\mathrm{d}x^j \wedge \mathrm{d}x^k.$$

We have to calculate $\mathrm{d}\alpha$, which we will do in components. We have:

$$[\mathrm{d}\alpha]_{ij} = \frac{1}{2!}\nabla_{[i}\alpha_{j]}.$$

extra info: $F = dA$ is the curvature for the connection A

be patient!

Since there is no curvature yet, all the $\Gamma = 0$ and we have $\nabla_i = \partial_i$. (I know ... I just could not resist.) We need to include the derivative with respect to time. In order to avoid all the c and write it more compact we will use a Greek index here.

$$\mathrm{d}\alpha = \frac{1}{2} E_{i,\mu}\, \mathrm{d}x^\mu \wedge \mathrm{d}x^i \wedge \mathrm{d}x^0 + \frac{1}{4} \epsilon_{ijk} B^i_{\ ,\mu}\, \mathrm{d}x^\mu \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k$$

We need to split up those implied sums over μ into temporal and spatial parts in order to group them.

$$= \frac{1}{2} E_{i,0}\, \mathrm{d}x^0 \wedge \mathrm{d}x^i \wedge \mathrm{d}x^0 + \frac{1}{2} E_{i,m}\, \mathrm{d}x^m \wedge \mathrm{d}x^i \wedge \mathrm{d}x^0 + \frac{1}{4} \epsilon_{ijk} B^i_{\ ,0}\, \mathrm{d}x^0 \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k + \frac{1}{4} \epsilon_{ijk} B^i_{\ ,m}\, \mathrm{d}x^m \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k$$

Summand 1: There is a form wedged together with itself, this term is just zero. Summand 3: We to anticommutations such that the temporal form is the very last one.

$$= \frac{1}{2} E_{i,m}\, \mathrm{d}x^m \wedge \mathrm{d}x^i \wedge \mathrm{d}x^0 + \frac{1}{4} \epsilon_{ijk} B^i_{\ ,0}\, \mathrm{d}x^j \wedge \mathrm{d}x^k \wedge \mathrm{d}x^0 + \frac{1}{4} \epsilon_{ijk} B^i_{\ ,m}\, \mathrm{d}x^m \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k$$

Summand 1 & 2: We factor out the last form and rename the dummy indices.

$$= \frac{1}{2} \left[E_{j,i}\, \mathrm{d}x^i \wedge \mathrm{d}x^j + \frac{1}{2} \epsilon_{lij} B^l_{\ ,0}\, \mathrm{d}x^i \wedge \mathrm{d}x^j \right] \wedge \mathrm{d}x^0 + \frac{1}{4} \epsilon_{ijk} B^i_{\ ,m}\, \mathrm{d}x^m \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k$$

Now we can factor out all the forms. To prepare the next step, we add a (now pointless) bracket in the second summand.

$$= \frac{1}{2} \left[E_{j,i} + \frac{1}{2} \epsilon_{lij} B^l_{\ ,0} \right] \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^0 + \frac{1}{4} \left[\epsilon_{ijk} B^i_{\ ,m} \right] \mathrm{d}x^m \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k$$

no! only its antisymm. part! $X_{ij}\, dx^i \wedge dx^j = 0 \Rightarrow X_{[ij]} = 0$, $X_{(ij)} \neq 0$

When we set this to zero, both brackets have to vanish individually. This will give us 9 equations for the first and 27 for the second bracket. Since they are antisymmetrized by the forms, we can add a Levi-Civita symbol to them and reduce the number of equations to 3 and 1 respectively. The $9+27$ equations are:

$$\cancel{E_{j,i} + \frac{1}{2} \epsilon_{lij} B^l_{\ ,0} = 0} \qquad \cancel{\epsilon_{ijk} B^i_{\ ,m} = 0}$$

And then we have

$$\epsilon^{ijk} E_{j,i} + \frac{1}{2} \epsilon^{ijk} \epsilon_{lij} B^l_{\ ,0} = 0 \qquad \epsilon^{jkm} \epsilon_{ijk} B^i_{\ ,m} = 0$$

The first term is the kth component of the curl in three dimensions. We contract the two Levi-Civita symbols in the second term and in the second equation.

$$[\operatorname{curl} \boldsymbol{E}]^k + \delta^k_l B^l_{\ ,0} = 0 \qquad 2\delta^m_i B^i_{\ ,m} = 0$$

Contracting the indices and dropping scalar factors, we get:

$$[\operatorname{curl} E]^k + \dot{B}^k = 0 \qquad \operatorname{div} B = 0.$$

And we can look at all the three equations at the same time and drop the indices.

$$\operatorname{curl} \boldsymbol{E} + \dot{\boldsymbol{B}} = 0 \qquad \operatorname{div} \boldsymbol{B} = 0$$

These are the four (or two if you count vector equations as one) homogen~~i~~eous Maxwell equations. Since those follow from $\mathrm{d}F = 0$ with F being the standard field strength 2-form, we think that we simply have $\alpha = F$ here. ✓

1.b. Ampère's law and Gauss's law 6/6

We can identify $-H_i \triangleq E_i$ and $D^k \triangleq B^k$ in the above calculation and directly give $\mathrm{d}\beta$:

$$\mathrm{d}\beta = \left[-H_{j,i} + \frac{1}{2}\epsilon_{lij} D^l_{\ ,0}\right] \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^0 + \frac{1}{2}\epsilon_{ijk} D^i_{\ ,m}\, \mathrm{d}x^m \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k$$ ✓

We can now add the other form to that.

$$\begin{aligned}\mathrm{d}\beta + 4\pi\gamma = &\left[-H_{j,i} + \frac{1}{2}\epsilon_{lij} D^l_{\ ,0}\right] \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^0 + \frac{1}{2}\epsilon_{ijk} D^i_{\ ,m}\, \mathrm{d}x^m \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k \\ &+ \frac{4\pi}{c}\epsilon_{ijk} J^k\, \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^0 - 4\pi\varrho\, \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z\end{aligned}$$

We regroup this by the forms again. For the $D^i_{\ ,m}$ term we use perform the antisymmetrization by hand by adding another Levi-Civita symbol.

$$\begin{aligned}= &\left[-H_{j,i} + \frac{1}{2}\epsilon_{lij} D^l_{\ ,0} + \frac{4\pi}{c}\epsilon_{ijk} J^k\right] \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^0 \\ &+ \left[\frac{1}{2}\epsilon^{mjk}\epsilon_{ijk} D^i_{\ ,m} - 4\pi\varrho\right] \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z\end{aligned}$$

We simplify the second term, it gives another divergence like before.

$$\begin{aligned}= &\left[-H_{j,i} + \frac{1}{2}\epsilon_{lij} D^l_{\ ,0} + \frac{4\pi}{c}\epsilon_{ijk} J^k\right] \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^0 \\ &+ [\operatorname{div} \boldsymbol{D} - 4\pi\varrho]\, \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z\end{aligned}$$

One equation then is $\operatorname{div} \boldsymbol{D} = 4\pi\varrho$. The other one (or ~~rather, nine~~ three) equations has to be simplified according to the same procedure as before.

$$\cancel{-H_{j,i} + \frac{1}{2}\epsilon_{lij} D^l_{\ ,0} + \frac{4\pi}{c}\epsilon_{ijk} J^k} = 0$$

~~Since those nine equations they are antisymmetric in i and j we can rewrite them as three unique equations, just like before.~~

$$-\epsilon^{ijm}H_{j,i} + \frac{1}{2}\epsilon^{ijm}\epsilon_{lij}D^l_{,0} + \frac{4\pi}{c}\epsilon^{ijm}\epsilon_{ijk}J^k = 0$$
$$-\epsilon^{ijm}H_{j,i} + \delta^m_l D^l_{,0} + \frac{8\pi}{c}\delta^m_k J^k = 0$$
$$-[\operatorname{curl} H]^m + D^m_{,0} + \frac{8\pi}{c}J^m = 0$$
$$-\operatorname{curl} H + \frac{1}{c}\dot{D} + \frac{8\pi}{c}J = 0$$

This can be rewritten in the usual form and gives:

4π ?

$$\operatorname{curl} H - \frac{1}{c}\dot{D} = \frac{8\pi}{c}J$$

That is the correct result in Gaussian units. (✓)

1.c. Continuity equation 2/6

→ all of the equations give a physical result!

~~The equation $d\beta + 4\pi\gamma = 0$ gives a physical result.~~ Applying the exterior derivative again on it gives ~~up~~ us $d\gamma = 0$.

Using the techniques used in the previous problems we can also use the definition of γ and derive that

$$\cancel{d\gamma = \operatorname{div} J + \dot{\varrho}}.$$

that would mean 4-form = 0-form !

Setting this to zero gives the known continuity equation which tells us that charge is conserved, ~~especially over time.~~ $\int_\Sigma d\gamma$ then gives the amount of charge gained or lost in a time interval dt. ✓

→ show this!

1.d. Relation of gauge freedom to Poincaré's lemma 2/6

You have shown in the exercise class on 2015-05-18 that the difference between two local vector potentials on a non-contractable space (like the sphere) can be rewritten as a gauge transformation of the form A (we just try to avoid to write "*vector* potential 1-*form*"). ✓

→ show the form $\vec{B} = \vec{\nabla}\times\vec{A}$, $\vec{E} = \vec{\nabla}A_0 + \partial_t\vec{A}$!

2. Cohomology groups of a torus

2.a. Zeroth cohomology group 4/4

We are only looking at 0-forms here, plain functions. If they are closed, this means that

$$df = f_{,i}\, dx^i = 0.$$ ✓

Scalar fields where all partial derivatives vanish have only one basis, $f(x, y) = 1$. They are not particularly fancy, just constant fields. These forms a vector space over **R**, such that $Z^0(T^2) = \mathbf{R}$. Since there are no -1-forms, $B^0(T^2) = \varnothing$ as given in the hint.

$\rightarrow x \sim y$ iif $x = y + 0$ $\Rightarrow$ id. map

We are not sure what it means to form the quotient set with the empty set as the divisor. It probably is the same as dividing through the set that only contains the identity, $\{e\}$. If that is the case, we have

$$H^0(T^2) = \frac{Z^p(T^2)}{\varnothing} = \mathbf{R}.$$

The set $H_0(T^2)$ is given as **Z** and later generalized to **R** in the lecture notes. Due to the connections between H_p and H^p, it seems that our result is in the right direction.

2.b. Integrals and first Cohomology group

6/6

Integrals To integrate this, we should probably use a pullback onto the curve that we are interested in. Since those lines are not really fancy, we might get away with doing it right there in the coordinates θ and ϕ. We can use the fundamental theorem of exterior calculus to compute those integrals. The boundary of the curves are just the points 0 and 2π with orientations $-$ and $+$, respectively.

$$\int_a \frac{\mathrm{d}\theta}{2\pi} = \int_{\partial a} \frac{\theta}{2\pi} = \frac{1}{2\pi}[-0 + 2\pi] = 1.$$

The second integral gives 1 by the exact same argument.

For the mixed integrals, we chose the curves a and b such that they go through the origin. Well, since we identity the boundaries all four of them go through the origin. We will use the ones where one of $\theta = 0$ or $\phi = 0$ holds. Then we can apply the Stokes's theorem like above and see that those integrals are zero:

$$\int_a \frac{\mathrm{d}\phi}{2\pi} = \int_{\partial a} \frac{\phi}{2\pi} = \frac{1}{2\pi}[-0 + 0] = 0.$$

First cohomology group The integrals tell us that the forms $\mathrm{d}\theta$ and $\mathrm{d}\phi$ are not exact *globally*. They are closed since they are exterior derivatives themselves. The functions θ and ϕ have a jump at 0, which causes them to be "inexact" globally. The 2-torus is not contractible, so it is no surprise that there are inexact closed forms on it.

The 1-forms that are exact globally have to come from the exterior derivative of scalar fields that are smooth and do not have a jump anywhere. Those could be expanded into a Fourier series in θ and ϕ with products of Fourier coefficients. One probably has to limit this to a finite number of coefficients, otherwise a sawtooth like the scalar field θ would be allowed again.

The set of exact forms can be build up from all possible smooth scalar fields f:

$$B^1(T^2) = \{f_{,i}(\theta, \phi)\,\mathrm{d}\Theta^i : f \in C^1(T^2)\},$$

where C^ω is the set of smooth functions that can be differentiated ω times.

The set of closed 1-forms is really limited, they have to be linear functions. A general 1-form would be

$$\omega = \omega_i(\theta, \phi)\,\mathrm{d}\Theta^i.$$

Since it has to be closed, we get the following condition:

$$\mathrm{d}\omega = \omega_{i,j}\,\mathrm{d}\Theta^j \wedge \mathrm{d}\Theta^i = \left[\omega_{2,1} - \omega_{1,2}\right]\mathrm{d}\theta \wedge \mathrm{d}\phi = 0 \iff \omega_{2,1} = \omega_{1,2}.$$

We define this to be $\lambda := \omega_{1,2}$. This λ also is $[\mathrm{d}\omega]_{12} = [\mathrm{d}\omega]_{21}$. Forms have to be antisymmetric, so only the antisymm. part ~~this has to be zero, then~~. We are looking for 1-forms where we have $\cancel{\omega_{i,j}}\ \omega_{[i,j]} = 0$, meaning that ω_i are just constant numbers. We can use this to build up a general closed 1-form:

$$Z^1(T^2) = \{c\,\mathrm{d}\theta + d\,\mathrm{d}\phi + \mathrm{d}f(\phi, \theta) : c, d \in R, f \in C^1(T^2)\}.$$

The conjugacy classes with respect to B are defined by the two real numbers $(c, d) \in \mathbf{R} \oplus \mathbf{R}$. We remove the freedom in the f and therefore just have

$$H^1(T^2) = \{c\,\mathrm{d}\theta + d\,\mathrm{d}\phi : c, d \in R\},$$

where we have chose then representative of the conjugacy class such that $f = 0$. We therefore have $H^1(T^2) \simeq \mathbf{R}^2$.

The lecture notes give $\cancel{H_2(T^2)}$ $H_1(T^2)$ to be $\mathbf{R}^2$, which is also our result here. Our result should therefore be correct, ~~perhaps for the wrong reasons, though.~~

2.c. Second cohomology group and closed forms 2/6

Already closed forms Since the two dimensional torus is a two dimensional space, there are no 3-forms. Every top dimensional form ~~are~~ is closed ~~in this sense~~.

Second cohomology group We are supposed to use that $H_2(T^2; \mathbf{R}) = \mathbf{R}$. This means that $H(T^2) \simeq \mathbf{R}$ as well. The 2-forms on a 2-dimensional space just have one free component. So $\omega_{12}(\theta, \phi)$ can be any smooth function on T^2. So it seems that $Z^2(T^2) \simeq C^0(T^2; \mathbf{R}^2)$

The exact forms have to come from $\mathrm{d}\alpha$, where the components of α are two smooth functions. Since the form $\mathrm{d}\alpha$ has to be antisymmetric, one has the condition $\alpha_{1,2} = -\alpha_{2,1}$. This looks a lot like the Cauchy-Riemann equation for analytic complex functions. These equations have to be fulfilled by a pair of functions on order to serve as real and imaginary part of an analytic function. We think this means that $B^2(T^2) \simeq C^0(T^2; \mathbf{C})$

Not every smooth function of two variables can be thought of as an analytic function. Analytic functions have the property that the real and imaginary parts are harmonic ($\Delta u = \Delta v = 0$), but we already knew that since they are top dimensional forms.

We have a hunch that B contains less elements than Z, but we do not really see why the quotient group should be isomorphic to $\mathbf{R}$.

e.g. $\omega = \mathrm{d}\theta \wedge \mathrm{d}\varphi$, $\mathrm{d}\omega = 0 \Rightarrow \omega \in Z$
but $\omega \neq \mathrm{d}\alpha \Rightarrow \omega \notin B$

3. Homotopy type of a punctured plane 10/10

Since we are going to write it a lot, we define the punctured plane to be $P := \mathbf{R}^2 \setminus \{0\}$.

We have the combined map:

$$\begin{aligned} i \circ r: \quad P &\to P \\ x &\mapsto \frac{x}{\|x\|}. \end{aligned}$$

To show that this map is homotopic to the identity on the space P, we have to find one smooth map F which satisfies the requirements from the definition of the homotopy. ✓

One such map would be given by

$$F(x,t) = \frac{x}{t\|x\| + 1 - t}. \quad ✓$$

It seems to be smooth everywhere in P and gives $i \circ r$ for $t = 1$ and the identity for $t = 0$.

Another map would be the linear interpolation between the two maps:

$$F(x,t) = t\frac{x}{\|x\|} + [1-t]x. \quad ✓$$

This also looks smooth.

Since there exists at least one mapping F, the two maps are homotopic to each other. ✓

$$\Rightarrow \boxed{P \cong S^1}$$