Disclaimer

This is a reviewed problem set for the module geometry.

This problem set was reviewed by a tutor. *This does not mean that it is a perfect solution. Neither I or the tutor imply that there are no further mistakes in this document.*

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[disclaimer]

Geometry in Physics

Problem Set 3

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Group 1 – Jens Boos

problem number	achieved points	possible points
1	13	~~20~~ 15
2	23	~~20~~ 25
3	9	10
Total	45	50

1. Alternating forms

1.a. Vector space 3/5

The forms we consider here are still linear by the definition on the problem set. Since ϕ is linear in all of its arguments, the definition of addition and scalar multiplication is straightforward. Since we consider the whole space of those alternating forms, the space is also closed under those operations. It must be a vector space. what is the neutral and inverse element?

1.b. Small p 5/5

In the definition of the space of alternating forms, the number p said how many elements of V are needed for the from to produce a real value from the field $\mathbf{R}$. So for $p = 0$, the resulting space is $\Lambda^0(V^*)$ which contains all the forms that do not take any element from V to produce a scalar. These are all the elements of $\mathbf{R}$ itself. nice explanation ✓

Now for $p = 1$ one needs to give the 1-form ϕ one element from V to get a scalar. Since it only has one argument, it cannot be not antisymmetric in it. So *any* 1-form is antisymmetric in this sense. ✓ Therefore all of V^* is allowed and we have $\Lambda^1(V^*) = V^*$. ✓

see also: generalized Kronecker symbol

1.c. Large p 5/5

Here we have an n-dimensional vector space and consider p-forms with $p > n$ on that. The problem is that there are only n independent vectors in an n-dimensional space. Since it is multilinear the whole thing can be expressed in terms of the basis vectors $\{e_i\}$ of the vector space. The action of ϕ on a set of p vectors can be expanded in terms of basis vectors. We start with expression the vectors in the basis:

$$\phi(v_1, \dots, v_p) = \phi(v_1^i e_i, \dots, v_p^k e_k).$$

Now we can use the linearity in each argument of ϕ and pull all the scalars out.

$$= v_1^i \dots v_p^k \phi(e_i, \dots, e_k).$$

The only way a summand in the implied sum can give a non-zero contribution is that all the indices $i, \dots, k$ are pairwise different. Since $i, \dots, k \in \mathbf{N} \cap [1, n]$, this is not possible! The only *alternating* form that could be defined is zero.

2.e.
~~1.d.~~ Dimensionality 3/5

The dimensionality of a space is equal to the number of basis vectors. We will show here that the number of basis vectors of the space of alternating forms is $\binom{n}{p}$. We want p-forms, they therefore need to take p vectors as arguments and generate a scalar from them. Since they form a vector space, every form can be written as a linear combination of basis elements. The forms are contained in the space $\otimes^p V$. The basis elements of this tensor space can be written as

$$\bigotimes_{k=0}^{p} e^{i_k}.$$

where the $\{i_k\}$ are out of $\mathbf{N} \cap [1, n]$ but not necessarily unique at this point. In other words, one has an n-dimensional space where one wants to create a linear mapping that takes p vectors as arguments. Those multilinear mappings can be written as products of simple linear mappings. To illustrate this with $p = 2$ here: Let $\phi(u, v) = \lambda$ be such a bilinear mapping. Then we can write this as

$$\phi(u, v) = u^i v^j \phi(e_i, e_j).$$

The n^2 components $\phi(e_i, e_j)$ completely describe the mapping. We can write ϕ in components as

$$\phi = \phi_{ij} e^i \otimes e^j.$$

good

Now ϕ can be thought of as a (0, 2) tensor which has n^2 components. Now comes the restriction part: Since this tensor has to be completely antisymmetric in all its indices, there is just one degree of freedom left, the magnitude of the ϕ_{12} component. Therefore, with $n = p = 2$, it actually is $\binom{2}{2}$. To generalize this to $n = 3$ but keeping $p = 2$ the basis vectors are now the following:

$$e^1 \otimes e^2, \quad e^2 \otimes e^3, \quad e^3 \otimes e^1$$

So there are three basis elements for two-forms in a three dimensional space. This is because there are $\binom{n}{p}$ ways of selecting p unique elements of a set with cardinality n. The $\phi_{ij\ldots k}$ does not introduce any more degrees of freedom as shown above.

basis for $\Lambda^p(V)$ is $\{e^{i_1} \wedge e^{i_2} \wedge \ldots \wedge e^{i_p}\}$, $i_1 < i_2 < \ldots < i_p$.
you come to $\binom{n}{p}$ a little too fast, more explanation is needed.

2. Exterior algebra

2.a. Maximum order

5/5

n is not defined again in this problem, but we will just assume it is the dimensionality of V just like before. And the second ϕ probably is a ψ, otherwise it would be defined twice. (you probably had an earlier version of the sheet)

ψ already is an alternating form with maximum order on the space V. Adding another order to it (and this is going to happen since $p > 0$) will make it identical to zero since there is no alternating form with $p > n$ as shown earlier.

This can also be shown in index notation. We use antisymmetrization notation by ~~Penrose (2005)~~ in the idempotent form which looks like this:

goes back to Schouten (1950s) and Bach

$$\phi_{[ij]k} := \frac{1}{2!}\left[\phi_{ijk} - \phi_{jik}\right].$$

Now Equation (3) from the problem set becomes:

$$[\phi \wedge \psi](v_1, \ldots, v_p, v_{p+1}, \ldots, v_{p+n}) = \frac{[p+n]!}{p!n!} \phi(v_{[1}, \ldots, v_p)\psi(v_{p+1}, \ldots, v_{p+n]}).$$

why square brackets?

The factor $[p+q]!$ is needed by my choice of idempotent antisymmetrization. Now we can expand each vector v_i in terms of its components and basis vectors and define each form in terms of its basis elements as well. This needs the introduction of another set of indices to sum over. The antisymmetrization stays at the vector components in this step.

$$= \frac{[p+n]!}{p!n!} \phi_{i\ldots j} \psi_{k\ldots l} v^i_{[1} \cdots v^j_p v^k_{p+1} \cdots v^l_{p+n]}$$

We can now move the antisymmetrization to the forms ϕ and ψ. Here we have a notational challenge: The indices are now multiple symbols, so we could delimit them with commas. But commas in indices denote partial derivatives. We will just leave it like this and hope the intent is clear.

$$= \frac{[p+n]!}{p!n!} \phi_{[1\ldots p} \psi_{p+1\ldots p+n]} v^i_1 \cdots v^j_p v^k_{p+1} \cdots v^l_{p+n}$$

The expression $\phi_{[1\ldots p}\psi_{p+1\ldots p+n]}$ now denotes a completely antisymmetric $p+q$ dimensional form. The problem is that this is in an n dimensional vector space which does not work: The antisymmetrization will pick up at least one index twice and therefore create pairs of terms with opposing signs. Those cancel each other and all is left is

$$= 0.$$

Once the forms are expressed with indices, the Equation (3) from the problem set can be written shorter as:

$$[\phi \wedge \psi]_{i\ldots kl\ldots m} = \frac{[p+n]!}{p!n!}\phi_{[i\ldots k}\psi_{l\ldots m]}.$$

2.b. Bilinearity 5/5

In the above equation, there is a simple product of two quantities, so it has to be bilinear:

$$\left[[\phi + c\tilde{\phi}] \wedge \psi\right]_{i\ldots kl\ldots m} = \frac{[p+n]!}{p!n!}[\phi + c\tilde{\phi}]_{[i\ldots k}\psi_{l\ldots m]}$$ ✓

Since the forms themselves are in a vector space, we can assign the indices to each one. The antisymmetrization notation breaks there, though. So we then also expand the product and obtain

$$= \frac{[p+n]!}{p!n!}\phi_{[i\ldots k}\psi_{l\ldots m]} + c\tilde{\phi}_{[i\ldots k}\psi_{l\ldots m]}.$$

Now we can pick up the shards and write this more compactly as

$$= \phi \wedge \psi + c\tilde{\phi} \wedge \psi.$$ ✓

dot?

2.c. Associativity 5/5

We have just shown the bilinearity by reducing the wedge product to a product of two simple numbers. For those, the associativity also holds. ✓

2.d. Graded commutativity 5/5

Look at

$$[\phi \wedge \psi]_{i\ldots kl\ldots m} = \frac{[p+n]!}{p!n!}\phi_{[i\ldots k}\psi_{l\ldots m]}$$

again. In order to swap them both, we need to swap the indices $i\ldots k$ with the set $l\ldots m$. For each index of the last set (q of them) we need to swap them through all the indices of the first set (p of them). Each swapping will change the sign because the forms are antisymmetric. In total there are pq swappings needed, therefore the total sign change is $[-1]^{pq}$. ✓

Can you explain to me why you use a square bracket sometimes?

3. Directional derivative

3.a. Concrete values 1|2

We have

$$p(t) = \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}, \quad v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and the function $f(p) = \frac{1}{2} p \cdot p$. We first compute the partial derivatives of the function f:

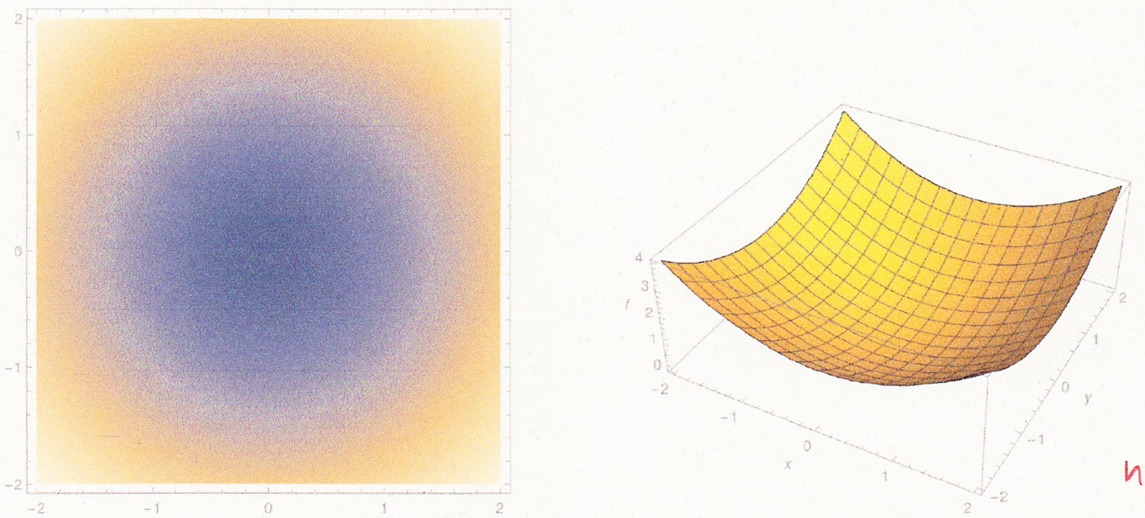
$$f_{,i}(p) = p_i. \quad \text{no metric needed, } f: V \to \mathbb{R}, \text{ not } V \times V \to \mathbb{R}.$$

Now contract this with the vector v and get

$$\mathrm{D}_{p,v}(f) = v^i p_i. = \cos(t)$$

3.b. Sketches 2|2

The plots of f are shown in Figure 1. Figure 2 shows the gradient $\mathrm{d}f$ and $v = p(0)$.



nice ✓

Figure 1: Visualizations of $f(x, y)$. The left one shows a good representation of f as a member $C^\infty(\mathbb{R}^2, \mathbb{R})$ since it the value at each point is a scalar and the domain is a two dimensional surface. The right side shows a 3D plot of the same function. The problem with this visualization is that it suggests that f embedded in $\mathbb{R}^3$ or something similar~~ly~~. It is better to think intrinsically to get beyond submanifolds. Created with *Mathematica* 10.

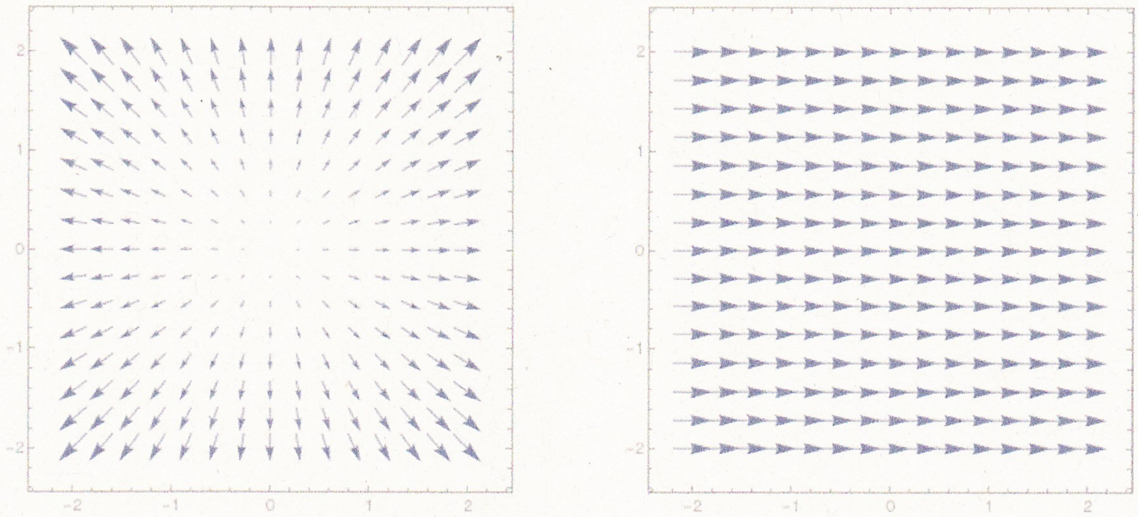


Figure 2: The left side shows ∂f. Either the coordinates are (e^1, e^2) and this is the actual gradient $\mathrm{d}f$, which is a 1-form. Or the axes show (e_1, e_2) and the plot shows the vector field $[\mathrm{d}f]^\sharp$. Since we are working in $\mathbf{R}^2$, there probably is an Euclidean metric and this can be done trivially. The right side visualizes v, which is an actual vector. It is a rather boring vector field since it is constant everywhere. $p(t)$ looks just the same, except that it is rotated by the angle t (in radians, of course). Sorry for the rastered images, *Mathematica 10* had some problem exporting those as PDF or EPS files.

3.c. Geometric interpretation 2/2

A geometric quantity does not depend on the basis. So the quantity interpreted must also be basis independent. With the notation usually used we would write

$$\mathrm{D}_{p,v}(f) = v \cdot [\nabla f](p).$$

sloppy. You mean vector calculus, right?

In exterior calculus: $D_{p,v}(f) = (df)(v)|_p$

So $\mathrm{D}_{p,v}(f)$ is the rate of change of f along v at the point p.

3.d. Derivation 4/4

Linearity The linearity in the argument of D is inherited from the linearity of the partial derivative and the linearity of v.

Leibniz law This one is more interesting. It follows from the Leibniz law of the partial derivative itself. By definition we have

$$\mathrm{D}_{p,v}(f\,g) = v^i [f\,g]_{,i}(p).$$

Now the partial derivative obeys the Leibniz law.

$$= v^i [f_{,i} g + f\,g_{,i}](p)$$

The function evaluation (at p) has to be applied to every single function.

$$= v^i \left[f_{,i}(p) g(p) + f(p) g_{,i}(p) \right]$$

We expand the bracket and let v act on each term individually.

$$= v^i f_{,i}(p) g(p) + v^i f(p) g_{,i}(p)$$

Since f and g produce scalars, we can move the terms around like we wish.

$$= v^i f_{,i}(p) g(p) + f(p) v^i g_{,i}(p)$$

Now we can recognize the definition of the derivation and insert it back again.

$$= \mathrm{D}_{p,v}(f) g(p) + f(p) \mathrm{D}_{p,v}(g)$$

Then we are done.

References

Penrose, Roger (2005). *Road to Reality*. 1. New York: Alfred A. Knopf. ISBN: 0-679-45443-8.